

Exceptional Lie Algebras, SU(3) and Jordan Pairs

Part 2: Zorn-type Representations

Alessio Marrani¹ and Piero Truini²

¹ *Instituut voor Theoretische Fysica, KU Leuven,
Celestijnenlaan 200D, B-3001 Leuven, Belgium*

`alessio.marrani@fys.kuleuven.be`

² *Dipartimento di Fisica, Università degli Studi
via Dodecaneso 33, I-16146 Genova, Italy*

`truini@ge.infn.it`

ABSTRACT

A representation of the exceptional Lie algebras is presented. It reflects a simple unifying view and it is realized in terms of Zorn-type matrices. The role of the underlying Jordan pair and Jordan algebra content is crucial in the development of the structure. Each algebra contains three Jordan pairs sharing the same Lie algebra of automorphisms and the same external $su(3)$ symmetry. Applications in physics are outlined.

Contents

1	Introduction	1
2	Jordan Pairs	6
3	Octonions	7
4	\mathfrak{g}_2 action on Zorn matrices	8
5	$\mathfrak{n} = 1$: Matrix representation of \mathfrak{f}_4	11
5.1	Comparison with Tits' construction	13
5.2	ϱ as a representation of \mathfrak{f}_4	14
6	$\mathfrak{n} = 2$: Matrix representation of \mathfrak{e}_6	16
7	$\mathfrak{n} = 4$: Matrix representation of \mathfrak{e}_7	19
8	Jacobi identity for $\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7$	21
9	$\mathfrak{n} = 8$: Matrix representation of \mathfrak{e}_8	24
10	Jacobi identity for \mathfrak{e}_8	26
11	Future developments	28

1 Introduction

Groups in physics have at least a threefold valence. First, they represent symmetries that, by definition, introduce elegance in all the equations which are manifestly symmetry invariant. Moreover, symmetries also arise as fundamental principles in constructing new theories, like, for example, gauge symmetries for the Standard Model (SM) of particle physics, conformal symmetry for string theory, or general covariance for the Einstein theory of relativity. Finally, symmetries - hence groups - play a key role in solving the equations of motion.

A particular class is represented by (semi)-simple Lie groups and algebras, which find application in a large number of mathematical and physical fields. All finite-dimensional complex Lie algebras have been classified by Killing, whose proofs have been made rigorous by Cartan, who has also extended the classification to non-compact real forms. This classification has led to the discovery, beyond the famous classical series, of five exceptional algebras together with the corresponding real forms: \mathfrak{g}_2 , \mathfrak{f}_4 , \mathfrak{e}_6 , \mathfrak{e}_7 and \mathfrak{e}_8 .

Exceptional Lie groups and algebras appear naturally as gauge symmetry groups of field theories which are low-energy limits of string models [1].

Various non-compact real forms of exceptional algebras occur in supergravity theories in different dimensions as U -duality¹. The related symmetric spaces are relevant by themselves for general relativity, because they are Einstein spaces [4]. In supergravity, some of these cosets, namely those pertaining to the non-compact real forms, are interpreted as scalar fields of the associated non-linear sigma model (see *e.g.* [5, 6], and also [7] for a review and list of Refs.). Moreover, they can represent the charge orbits of electromagnetic fluxes of black holes when the Attractor Mechanism [8] is studied ([9]; for a comprehensive review, see *e.g.* [10]), and they

¹Here U -duality is referred to as the “continuous” symmetries of [2]. Their discrete versions are the U -duality non-perturbative string theory symmetries introduced in [3].

also appear as the *moduli spaces* [11] for extremal black hole attractors; this approach has been recently extended to all kinds of branes in supergravity [12]. Fascinating group theoretical structures arise clearly in the description of the Attractor Mechanism for black holes in the Maxwell-Einstein supergravity, such as the so-called magic exceptional $\mathcal{N} = 2$ supergravity [13] in four dimensions, which is related to the minimally non-compact real $\mathfrak{e}_{7(-25)}$ form [14] of \mathfrak{e}_7 .

The smallest exceptional Lie algebra, \mathfrak{g}_2 , occurs for instance in the deconfinement phase transitions [15], in random matrix models [16], and in matrix models related to D -brane physics [17]; it also finds application to Montecarlo analysis [18].

\mathfrak{f}_4 enters the construction of integrable models on exceptional Lie groups and of the corresponding coset manifolds. Of particular interest, from the mathematical point of view, is the coset manifold $\mathbb{CP}^2 = F_4/Spin(9)$, the octonionic projective plane (see *e.g.* [19], and Refs. therein). Furthermore, the split real form $\mathfrak{f}_{4(4)}$ has been recently proposed as the global symmetry of an exotic ten-dimensional theory in the context of gauge/gravity correspondence and “magic pyramids” in [20].

Starting from the pioneering work of Gürsey [21, 22] on Grand Unified theories (GUTs), exceptional Lie algebras have been related to the study of the SM, and to the attempts to go beyond it: for example, the discovery of neutrino oscillations, the fine tuning of the mixing matrices, the hierarchy problem, the difficulty in including gravity, and so on. The renormalization flow of the coupling constants suggests the unification of gauge interactions at energies of the order of 1015 GeV, which can be improved and fine tuned by supersymmetry. In this framework the gauge group G of Grand Unified theory (GUT) is expected to be simple, to contain the SM gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ and also to predict the correct spectra after spontaneous symmetry breaking. The particular structure of the neutrino mixing matrix has led to the proposal of G given by the semi-direct product between the exceptional group E_6 and the discrete group S_4 [23].

Recently, \mathfrak{e}_7 and “groups of type E_7 ” [24] have appeared in several indirectly related contexts. They have been investigated in relation to minimal coupling of vectors and scalars in cosmology and supergravity [25]. They have been considered as gauge and global symmetries in the so-called Freudenthal gauge theory [26]. Another application is in the context of entanglement in quantum information theory; this is actually related to its application to black holes via the black-hole/qubit correspondence (see [27] for reviews and list of Refs.).

The largest finite-dimensional exceptional Lie algebra, namely \mathfrak{e}_8 , appears in supergravity [28] in its maximally non-compact (split) real form, whereas the compact real form appears in heterotic string theory [29]. Rather surprisingly, in recent times the popular press has been dealing with \mathfrak{e}_8 more than once. Firstly, the computation of the Kazhdan-Lusztig-Vogan polynomials [30] involved the split real form of \mathfrak{e}_8 . Then, attempts at formulating a “theory of everything” were considered in [31], but they were proved to be unsuccessful (*cfr. e.g.* [32]). More interestingly, the compact real form of \mathfrak{e}_8 appears in the context of the cobalt niobate ($CoNb_2O_6$) experiment, making this the first actual experiment to detect a phenomenon that could be modeled using \mathfrak{e}_8 [33].

It should also be recalled that alternative approaches to quantum gravity, such as loop quantum gravity, [34] have also led towards the exceptional algebras, and \mathfrak{e}_8 in particular (see *e.g.* [35]).

It is worth mentioning that the adjoint of \mathfrak{e}_8 is its smallest fundamental representation; this sets \mathfrak{e}_8 on a different footing with respect to all other Lie algebras for unifying theories, which all exhibit a fundamental representation of lower dimension than the adjoint - of dimension 7, 26, 27, 56, in particular, for the exceptional algebras \mathfrak{g}_2 , \mathfrak{f}_4 , \mathfrak{e}_6 , \mathfrak{e}_7 respectively. In the framework of a unified physical theory, therefore, only an \mathfrak{e}_8 -based model has matter particles, intermediate bosons, Higgs(es) *etc.* all in the same (adjoint, 248-dimensional) representation.

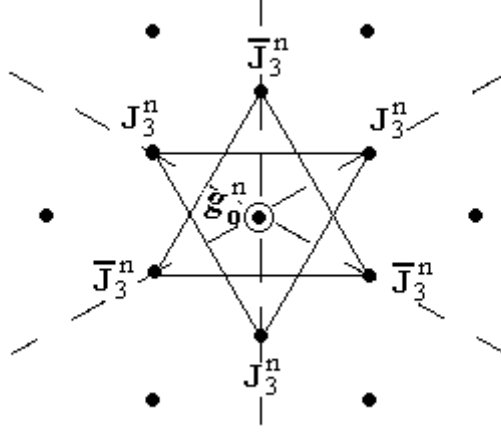


Figure 1: A unifying view of the roots of exceptional Lie algebras

There is a wide consensus in both mathematics and physics on the appeal of the largest exceptional Lie algebra \mathfrak{e}_8 , considered by many beautiful in spite of its complexity.

The present paper is the continuation of a previous one [36], in which the (finite-dimensional) exceptional Lie algebras were studied from a unifying point of view represented by the diagram in figure 1.

Figure 1 shows the projection of the roots of the exceptional Lie algebras on a complex $\mathfrak{su}(3) = \mathfrak{a}_2$ plane, recognizable by the dots forming the external hexagon, and it exhibits the *Jordan pair* content of each exceptional Lie algebra. There are three Jordan pairs $(\mathbf{J}_3^n, \bar{\mathbf{J}}_3^n)$, each of which lies on an axis symmetrically with respect to the center of the diagram. Each pair doubles a simple Jordan algebra of rank 3, \mathbf{J}_3^n , with involution - the conjugate representation $\bar{\mathbf{J}}_3^n$, which is the algebra of 3×3 Hermitian matrices over \mathbf{H} , where $\mathbf{H} = \mathbf{R}, \mathbf{C}, \mathbf{Q}, \mathbf{C}$ for $\mathbf{n} = 1, 2, 4, 8$ respectively, stands for real, complex, quaternion, octonion algebras, the four composition algebras according to Hurwitz's Theorem - see *e.g.* [37]. Exceptional Lie algebras $\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ are obtained for $\mathbf{n} = 1, 2, 4, 8$, respectively. \mathfrak{g}_2 can be also represented in the same way, with the Jordan algebra reduced to a single element; this corresponds to setting $\mathbf{n} = -2/3$; in Table 1 below. The Jordan algebras \mathbf{J}_3^n (and their conjugate $\bar{\mathbf{J}}_3^n$) globally behave like a $\mathbf{3}$ (and a $\bar{\mathbf{3}}$) dimensional representation of the outer \mathfrak{a}_2 . The algebra denoted by \mathfrak{g}_0^n in the center (plus the Cartan generator associated with the axis along which the pair lies) is the algebra of the automorphism group of the Jordan Pair; namely, \mathfrak{g}_0^n is the *reduced* structure group of the corresponding Jordan algebra \mathbf{J}_3^n : $\mathfrak{g}_0^n = \mathbf{str}_0(\mathbf{J}_3^n)$. Notice that \mathbf{J}_3^n fits into a $(3\mathbf{n} + 3)$ -dimensional irreducible representation of \mathfrak{g}_0^n itself.

The *base field* considered throughout the present paper is \mathbf{C} . Therefore, all parameters in the whole paper are *complex* numbers. For instance, \mathbf{J}_3^1 is a *real* Jordan algebra over the complex numbers, which means that the Hermitian conjugation is the transposed of matrices: \mathbf{J}_3^1 is an algebra of symmetric complex matrices.

The reason for choosing complex numbers as base field lies in the fact that we are dealing with the root diagrams of the Lie algebras, therefore we need an algebraically closed field. The various real compact and non-compact forms of the exceptional Lie algebras follow as a consequence, using some more or less laborious tricks, whose treatment we leave to a future study, and they do not affect the essential structure.

The *real, complex, quaternion, octonion* attributes corresponding to setting $\mathbf{n} = 1, 2, 4, 8$ in \mathbf{J}_3^n refer to algebras - over the complex field - whose role is that of a book-keeping device:

they are used in order to make the language easier and more compact. In this sense, they fall naturally into the Lie structure.

These algebras \mathbf{R} , \mathbf{C} , \mathbf{Q} , \mathfrak{C} are in general *non-commutative* (octonions - *Cayley numbers* - \mathfrak{C} are also *non-associative*), but they all are *alternative*, a fundamental property without which our whole construction would fall apart². They, however, have nothing to do with the base field - the complex field \mathbf{C} - of the corresponding Lie algebras. On the other hand, it is true the opposite : having complex alternative algebras allows to have nilpotents, which are as useful as J^+ and J^- are in the algebra of spin, or as creation and annihilation operators are in the description of the quantum harmonic oscillator or in quantum field theory.

By varying \mathbf{n} , figure 1 depicts the following decomposition, [36] :

$$\mathbf{L}^{\mathbf{n}} = \mathbf{a}_2 \oplus \mathbf{str}_0(\mathbf{J}_3^{\mathbf{n}}) \oplus \mathbf{3} \times \mathbf{J}_3^{\mathbf{n}} \oplus \bar{\mathbf{3}} \times \bar{\mathbf{J}}_3^{\mathbf{n}}, \quad (1.1)$$

with the corresponding compact cases given in Table 1:

\mathbf{n}	8	4	2	1	0	$-2/3$	-1
$\mathbf{L}^{\mathbf{n}}$	$\mathbf{e}_{8(-248)}$	$\mathbf{e}_{7(-133)}$	$\mathbf{e}_{6(-78)}$	$\mathbf{f}_{4(-52)}$	$\mathbf{so}(8)$	$\mathbf{g}_{2(-14)}$	$\mathbf{su}(3)$
$\mathbf{str}_{0,c}$	$\mathbf{e}_{6(-78)}$	$\mathbf{su}(6)$	$\mathbf{su}(3) \oplus \mathbf{su}(3)$	$\mathbf{su}(3)$	$\mathbf{u}(1) \oplus \mathbf{u}(1)$	—	—

Table 1: The exceptional sequence

The sequence $\mathbf{L}^{\mathbf{n}}$ is usually named “*exceptional sequence*” (or “*exceptional series*”; see *e.g.* [38], and Refs. therein). This can be either interpreted as a sequence of Lie algebras over the complex numbers \mathbf{C} , as we will consider throughout the present investigation, or as a sequence of corresponding *compact* real forms.

It is here worth pointing out that, by considering suitable non-compact, real forms, one obtains the \mathbf{n} -parametrized sequence of U -duality Lie algebras $\mathbf{L}^{\mathbf{n}}$ in $D = 3$ (Lorentzian) space-time dimensions³ [39] :

$$\mathbf{L}^{\mathbf{n}} = \mathbf{sl}(3, \mathbf{R}) \oplus \mathbf{str}_0(\mathbf{J}_3^{\mathbf{n}}) \oplus \mathbf{3} \times \mathbf{J}_3^{\mathbf{n}} \oplus \mathbf{3}' \times \mathbf{J}_3^{\mathbf{n}'}. \quad (1.2)$$

Note that the reduced structure Lie algebra $\mathbf{str}_0(\mathbf{J}_3^{\mathbf{n}})$, which, as stated above, is a suitable non-compact real form of $\mathbf{g}_0^{\mathbf{n}}$, is nothing but the $D = 5$ U -duality Lie algebra. Also,

$$\mathbf{L}^{\mathbf{n}} = \mathbf{qconf}(\mathbf{J}_3^{\mathbf{n}}) \quad (1.3)$$

is the *quasi-conformal* Lie algebra of $\mathbf{J}_3^{\mathbf{n}}$ [40, 41], *i.e.* the U -duality Lie algebra in $D = 3$ (see *e.g.* [42] and [10] for an introduction to the application of Jordan algebras and their symmetries in supergravity⁴, and lists of Refs.). Suitable real, non-compact forms of all exceptional Lie algebras can thus be characterized as *quasi-conformal* algebras⁵ of Euclidean simple Jordan algebras of rank 3.

²Non-alternative extensions beyond \mathfrak{C} , such as *sedenions* and *trigintaduonions* (*cfr. e.g.* [60]) would require a different approach.

³Jordan pairs of *semi-simple* Euclidean Jordan algebras of rank 3 in supergravity theories (among which the case of $\mathbf{so}(8)$, $\mathbf{n} = 0$) has been presented in [39].

⁴In these theories, the U -duality Lie algebra in $D = 4$ (Lorentzian) space-time dimensions is given by the *conformal* Lie algebra $\mathbf{conf}(\mathbf{J}_3^{\mathbf{n}}) = \mathbf{aut}(\mathfrak{F}(\mathbf{J}_3^{\mathbf{n}}))$, where $\mathfrak{F}(\mathbf{J}_3^{\mathbf{n}})$ denotes the Freudenthal triple system constructed over $\mathbf{J}_3^{\mathbf{n}}$.

⁵The case $\mathbf{n} = -1$ is trivial, and it corresponds to “*pure*” $\mathcal{N} = 2$ supergravity in four-dimensional Lorentzian space-time; therefore, it does not admit an uplift to five dimensions, and it will henceforth not be considered. Moreover, $\mathbf{su}(2)$ might be considered as the $\mathbf{n} = -4/3$ element of the sequence in Table below (1.1), as well. However, this is a limit case of the “exceptional” sequence reported in Table 1, not pertaining to Jordan pairs nor to supergravity in $D = 3$ dimensions, and thus we will disregard it.

At group level, the algebraic decompositions (1.1) and (1.2) are Cartan decompositions respectively pertaining to the following *maximal non-symmetric* embeddings:

$$QConf_c(\mathfrak{J}_3^q) \supset SU(3) \times Str_{0,c}(\mathfrak{J}_3^q); \quad (1.4)$$

$$QConf(\mathfrak{J}_3^q) \supset SL(3, \mathbf{R}) \times Str_0(\mathfrak{J}_3^q). \quad (1.5)$$

As mentioned above, the non-semi-simple part of the r.h.s. of (1.1) and (1.2) is given by a triplet of Jordan pairs.

Finally, we recall that in [39], by exploiting the Jordan pair structure of *U*-duality Lie algebras in $D = 3$ and the relation to the *super-Ehlers* symmetry in $D = 5$ [43], the massless multiplet structure of the spectrum of a broad class of $D = 5$ supergravity theories was investigated.

In general, many properties of Lie algebras and groups can be already inferred from abstract theoretical considerations; however, for most applications, it is useful to have explicit concrete realizations in terms of matrices⁶.

In this paper we develop the results of [36] and fully exploit Jordan pairs and the corresponding unifying view depicted in figure 1. We introduce *Zorn-type* matrix realizations of all exceptional finite-dimensional Lie algebras, which make the Jordan pair structure manifest and are written in the form of a 2×2 matrix, endowed with a quite peculiar matrix product accounting for the complexity and non-associativity of the underlying structure. As a consequence of (1.3), this corresponds to the explicit construction of *Zorn-type* matrix realizations of the compact form of quasi-conformal algebras of simple Jordan algebras of rank 3; we point out that in the present paper we will deal with Lie algebras over \mathbb{C} , leaving the analysis of real forms to future investigation.

The paper is organized as follows.

In section 2 we briefly review the concept of a Jordan pair. Most of the section can be found also in [36] and is repeated here for completeness.

For the same reason, as well as for introducing some notation, we present in section 3 a summary on the octonion algebra and its representation through the Zorn matrices, on which we base the development of our representations. The key idea which we exploit here is that the octonions' non-associativity can be cast into a properly defined product of 2×2 complex matrices.

With this in mind, we are able to define, formally using 2×2 matrices, a representation of \mathfrak{g}_2 in section 4, \mathfrak{f}_4 in section 5 (where we also make a comparison with Tits' construction), \mathfrak{e}_6 in section 6, \mathfrak{e}_7 in section 7. In section 8 we prove the Jacobi identity for all these algebras.

In the case of \mathfrak{e}_8 , section 9, a new difficulty occurs due to non-associativity. Not only the octonions are non-associative, but so is the underlying standard matrix product of the Jordan algebra elements. This forces a new definition of matrix elements and of their product, which still allows us to formally describe the representation of \mathfrak{e}_8 through 2×2 matrices. The proof of the Jacobi identity for this case heavily relies on the Jordan Pair axioms, and it is presented in section 10.

The paper ends with some proposals of future developments of the present work.

⁶Explicit realizations of exceptional groups have been obtained *e.g.* in [44]. Our results, however, displays a much more manageable form, with manifest \mathfrak{a}_2 covariance, as a consequence of the full exploitation of the underlying Jordan pair structure.

2 Jordan Pairs

In this section we review the concept of a Jordan Pair, [45] (see also [37] for an enlightening overview).

Jordan Algebras have traveled a long journey, since their appearance in the 30's [46]. The modern formulation [47] involves a quadratic map $U_x y$ (like xyx for associative algebras) instead of the original symmetric product $x \cdot y = \frac{1}{2}(xy + yx)$. The quadratic map and its linearization $V_{x,y} z = (U_{x+z} - U_x - U_z)y$ (like $xyz + zyx$ in the associative case) reveal the mathematical structure of Jordan Algebras much more clearly, through the notion of inverse, inner ideal, generic norm, *etc.* The axioms are:

$$U_1 = Id \quad , \quad U_x V_{y,x} = V_{x,y} U_x \quad , \quad U_{U_x y} = U_x U_y U_x \quad (2.1)$$

The quadratic formulation led to the concept of Jordan Triple systems [48], an example of which is a pair of modules represented by rectangular matrices. There is no way of multiplying two matrices x and y , say $n \times m$ and $m \times n$ respectively, by means of a bilinear product. But one can do it using a product like xyx , quadratic in x and linear in y . Notice that, like in the case of rectangular matrices, there needs not be a unity in these structures. The axioms are in this case:

$$U_x V_{y,x} = V_{x,y} U_x \quad , \quad V_{U_x y, y} = V_{x, U_y x} \quad , \quad U_{U_x y} = U_x U_y U_x \quad (2.2)$$

Finally, a Jordan Pair is defined just as a pair of modules (V^+, V^-) acting on each other (but not on themselves) like a Jordan Triple:

$$\begin{aligned} U_{x^\sigma} V_{y^{-\sigma}, x^\sigma} &= V_{x^\sigma, y^{-\sigma}} U_{x^\sigma} \\ V_{U_{x^\sigma} y^{-\sigma}, y^{-\sigma}} &= V_{x^\sigma, U_{y^{-\sigma}} x^\sigma} \\ U_{U_{x^\sigma} y^{-\sigma}} &= U_{x^\sigma} U_{y^{-\sigma}} U_{x^\sigma} \end{aligned} \quad (2.3)$$

where $\sigma = \pm$ and $x^\sigma \in V^{+\sigma}$, $y^{-\sigma} \in V^{-\sigma}$.

Jordan pairs are strongly related to the Tits-Kantor-Koecher construction of Lie Algebras \mathfrak{L} [49]-[51] (see also the interesting relation to Hopf algebras, [52]):

$$\mathfrak{L} = J \oplus \text{str}(J) \oplus \bar{J} \quad (2.4)$$

where J is a Jordan algebra and $\text{str}(J) = L(J) \oplus \text{Der}(J)$ is the structure algebra of J [37]; $L(x)$ is the left multiplication in J : $L(x)y = x \cdot y$ and $\text{Der}(J) = [L(J), L(J)]$ is the algebra of derivations of J (the algebra of the automorphism group of J) [53][54].

In the case of complex exceptional Lie algebras, this construction applies to \mathfrak{e}_7 , with $J = \mathbf{J}_3^8$, the 27-dimensional exceptional Jordan algebra of 3×3 Hermitian matrices over the complex octonions, and $\text{str}(J) = \mathfrak{e}_6 \otimes \mathbf{C} - \mathbf{C}$ denoting the complex field. The algebra \mathfrak{e}_6 is called the *reduced structure algebra* of J , $\text{str}_0(J)$, namely the structure algebra with the generator corresponding to the multiplication by a complex number taken away: $\mathfrak{e}_6 = L(J_0) \oplus \text{Der}(J)$, with J_0 denoting the traceless elements of J .

We conclude this introductory section with some standard definitions and identities in the theory of Jordan algebras and Jordan pairs, with particular reference to \mathbf{J}_3^n , $n = 1, 2, 4, 8$. If $x, y \in \mathbf{J}_3^n$ and xy denotes their standard matrix product, we denote by $x \cdot y := \frac{1}{2}(xy + yx)$ the Jordan product of x and y . The Jordan identity is the power associativity with respect to this product:

$$x^2 \cdot (x \cdot z) - x \cdot (x^2 \cdot z) = 0, \quad (2.5)$$

Another fundamental product is the *sharp* product $\#$, [37]. It is the linearization of $x^\# := x^2 - t(x)x - \frac{1}{2}(t(x^2) - t(x)^2)I$, with $t(x)$ denoting the trace of $x \in \mathbf{J}_3^n$, in terms of which we may write the fundamental cubic identity for \mathbf{J}_3^n , $n = 1, 2, 4, 8$:

$$x^\# \cdot x = \frac{1}{3}t(x^\#, x)I \quad \text{or} \quad x^3 - t(x)x^2 + t(x^\#)x - \frac{1}{3}t(x^\#, x)I = 0 \quad (2.6)$$

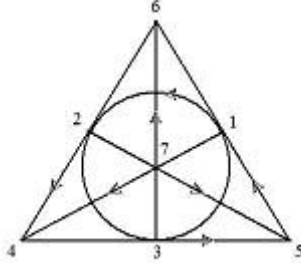


Figure 2: Fano diagram for the octonions' products

where we use the notation $t(x, y) := t(x \cdot y)$ and $x^3 = x^2 \cdot x$ (notice that for \mathbf{J}_3^8 , because of non-associativity, $x^2x \neq xx^2$ in general).

The triple product is defined as, [37]:

$$\begin{aligned} \{x, y, z\} := V_{x,y}z : &= t(x, y)z + t(z, y)x - (x \# z) \# y \\ &= 2[(x \cdot y) \cdot z + (y \cdot z) \cdot x - (z \cdot x) \cdot y] \end{aligned} \quad (2.7)$$

Notice that the last equality of (2.7) is not trivial at all. $V_{x,y}z$ is the linearization of the quadratic map U_xy . The equation (2.3.15) at page 484 of [37] shows that:

$$U_xy = t(x, y)x - x \# \# y = 2(x \cdot y) \cdot x - x^2 \cdot y \quad (2.8)$$

We shall make use of the following identities, which can be derived from the Jordan Pair axioms, [45]:

$$[V_{x,y}, V_{z,w}] = V_{V_{x,y}z, w} - V_{z, V_{x,y}w} \quad (2.9)$$

and, for $D = (D_+, D_-)$ a derivation of the Jordan Pair V and $\beta(x, y) = (V_{x,y}, -V_{y,x})$,

$$[D, \beta(x, y)] = \beta(D_+(x), y) + \beta(x, D_-(y)) \quad (2.10)$$

3 Octonions

As we introduced in Sec. 1, \mathfrak{C} stands for the algebra of the octonions (Cayley numbers) over the complex field \mathbf{C} whose multiplication rule goes according to the Fano diagram in figure 2.

If $a \in \mathfrak{C}$ we write $a = a_0 + \sum_{k=1}^7 a_k u_k$, where $a_k \in \mathbf{C}$ for $k = 1, \dots, 7$ and u_k for $k = 1, \dots, 7$ denote the octonion imaginary units. We denote by i the imaginary unit in \mathbf{C} .

Thence, we introduce 2 idempotent elements:

$$\rho^\pm = \frac{1}{2}(1 \pm iu_7)$$

and 6 nilpotent elements:

$$\varepsilon_k^\pm = \rho^\pm u_k, \quad k = 1, 2, 3$$

One can readily check that:

$$\begin{aligned}
(\rho^\pm)^2 &= \rho^\pm \quad , \quad \rho^\pm \rho^\mp = 0 \\
\rho^\pm \varepsilon_k^\pm &= \varepsilon_k^\pm \rho^\mp = \varepsilon_k^\pm \\
\rho^\mp \varepsilon_k^\pm &= \varepsilon_k^\pm \rho^\pm = 0 \\
(\varepsilon_k^\pm)^2 &= 0 \quad , \quad k = 1, 2, 3 \\
\varepsilon_k^\pm \varepsilon_{k+1}^\pm &= -\varepsilon_{k+1}^\pm \varepsilon_k^\pm = \varepsilon_{k+2}^\mp \quad (\text{indices modulo } 3) \\
\varepsilon_j^\pm \varepsilon_k^\mp &= 0 \quad j \neq k \\
\varepsilon_k^\pm \varepsilon_k^\mp &= -\rho^\pm \quad , \quad k = 1, 2, 3
\end{aligned} \tag{3.1}$$

It is known that octonions can be represented by Zorn matrices, [55]. If $a \in \mathfrak{C}$, $A^\pm \in \mathbb{C}^3$ is a vector with complex components α_k^\pm , $k = 1, 2, 3$ (and we use the standard summation convention over repeated indices throughout), then we have the identification:

$$a = \alpha_0^+ \rho^+ + \alpha_0^- \rho^- + \alpha_k^+ \varepsilon_k^+ + \alpha_k^- \varepsilon_k^- \longleftrightarrow \begin{bmatrix} \alpha_0^+ & A^+ \\ A^- & \alpha_0^- \end{bmatrix}; \tag{3.2}$$

therefore, through Eq. (3.2), the product of $a, b \in \mathfrak{C}$ corresponds to:

$$\begin{aligned}
&\begin{bmatrix} \alpha^+ & A^+ \\ A^- & \alpha^- \end{bmatrix} \begin{bmatrix} \beta^+ & B^+ \\ B^- & \beta^- \end{bmatrix} \\
&= \begin{bmatrix} \alpha^+ \beta^+ + A^+ \cdot B^- & \alpha^+ B^+ + \beta^- A^+ + A^- \wedge B^- \\ \alpha^- B^- + \beta^+ A^- + A^+ \wedge B^+ & \alpha^- \beta^- + A^- \cdot B^+ \end{bmatrix}, \tag{3.3}
\end{aligned}$$

where $A^\pm \cdot B^\mp = -\alpha_K^\pm \beta_K^\mp$ and $A \wedge B$ is the standard vector product of A and B .

4 \mathbf{g}_2 action on Zorn matrices

In this section, we derive the matrix representation of \mathbf{g}_2 , and its action on Zorn matrices. Let $a, b, c \in \mathfrak{C}$. Then the derivations of the octonions, [53] [56], can be written as $D_{a,b}$:

$$D_{a,b}c = \frac{1}{3}[[a, b], c] - (a, b, c) \quad \text{where } (a, b, c) = (ab)c - a(bc)$$

We choose the following \mathbf{g}_2 generators, for $k = 1, 2, 3 \pmod{3}$:

$$\begin{aligned}
d_k^\pm &= \mp D_{\varepsilon_{k+1}^\pm, \varepsilon_{k+2}^\mp} = \mp L_{\varepsilon_{k+1}^\pm} L_{\varepsilon_{k+2}^\mp} \\
H_1 &= \frac{\sqrt{2}}{2} \left(D_{\varepsilon_1^-, \varepsilon_1^+} - D_{\varepsilon_2^-, \varepsilon_2^+} \right) = \frac{\sqrt{2}}{2} \left(L_{\varepsilon_1^-} L_{\varepsilon_1^+} - L_{\varepsilon_2^-} L_{\varepsilon_2^+} \right) \\
H_2 &= \frac{\sqrt{6}}{6} \left(D_{\varepsilon_1^-, \varepsilon_1^+} + D_{\varepsilon_2^-, \varepsilon_2^+} - 2D_{\varepsilon_3^-, \varepsilon_3^+} \right) \\
&= \frac{\sqrt{6}}{6} \left(L_{\varepsilon_1^-} L_{\varepsilon_1^+} + L_{\varepsilon_2^-} L_{\varepsilon_2^+} - 2L_{\varepsilon_3^-} L_{\varepsilon_3^+} \right) \\
g_k^\pm &= 3 D_{\rho^\pm, \varepsilon_k^\pm} = L_{\varepsilon_k^\pm} - R_{\varepsilon_k^\pm} - 3L_{\rho^\mp} L_{\varepsilon_k^\pm}
\end{aligned}$$

We notice that $D_{\rho^+, \rho^-} = 0$, $D_{\rho^+, \varepsilon_k^\pm} = -D_{\rho^-, \varepsilon_k^\pm} = \mp D_{\varepsilon_{k+1}^\mp, \varepsilon_{k+2}^\mp}$ and that $D_{\varepsilon_1^-, \varepsilon_1^+} + D_{\varepsilon_2^-, \varepsilon_2^+} + D_{\varepsilon_3^-, \varepsilon_3^+} = 0$, hence the 14 generators introduced above span all the derivations of \mathfrak{C} .

The action of these generators on $a \in \mathfrak{C}$, $a = \alpha_0^+ \rho^+ + \alpha_0^- \rho^- + \alpha_k^+ \varepsilon_k^+ + \alpha_k^- \varepsilon_k^-$ is:

$$\begin{aligned} d_k^\pm &: a \rightarrow \pm(\alpha_{k+2}^\pm \varepsilon_{k+1}^\pm - \alpha_{k+1}^\mp \varepsilon_{k+2}^\mp) \\ H_1 &: a \rightarrow \frac{\sqrt{2}}{2}(\alpha_1^+ \varepsilon_1^+ - \alpha_2^+ \varepsilon_2^+ - \alpha_1^- \varepsilon_1^- + \alpha_2^- \varepsilon_2^-) \\ H_2 &: a \rightarrow \frac{\sqrt{6}}{6}(\alpha_1^+ \varepsilon_1^+ + \alpha_2^+ \varepsilon_2^+ - 2\alpha_3^+ \varepsilon_3^+ - \alpha_1^- \varepsilon_1^- - \alpha_2^- \varepsilon_2^- + 2\alpha_3^- \varepsilon_3^-) \\ g_k^\pm &: a \rightarrow -\alpha_{k+1}^\pm \varepsilon_{k+2}^\mp + \alpha_{k+2}^\pm \varepsilon_{k+1}^\mp - \alpha_k^\mp(\rho^\pm - \rho^\mp) - (\alpha_0^\pm - \alpha_0^\mp) \varepsilon_k^\pm \end{aligned}$$

One can thus readily check that $[H_1, H_2] = 0$ and that the g_k^\pm 's are eigenvectors of (H_1, H_2) , with respect to the Lie product, with eigenvalues $(\pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{6}}{6})$ and $(0, \pm \frac{\sqrt{6}}{3})$; the same with for d_k^\pm 's with eigenvalues $(\pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{6}}{2})$ and $(\pm \sqrt{2}, 0)$, namely :

$$\begin{aligned} [H_1, H_2] &= 0 \\ [H_1, g_1^\pm] &= \pm \frac{\sqrt{2}}{2} g_1^\pm & [H_2, g_1^\pm] &= \pm \frac{\sqrt{6}}{6} g_1^\pm & [H_1, d_1^\pm] &= \mp \frac{\sqrt{2}}{2} d_1^\pm & [H_2, d_1^\pm] &= \pm \frac{\sqrt{6}}{2} d_1^\pm \\ [H_1, g_2^\pm] &= \mp \frac{\sqrt{2}}{2} g_2^\pm & [H_2, g_2^\pm] &= \pm \frac{\sqrt{6}}{6} g_2^\pm & [H_1, d_2^\pm] &= \mp \frac{\sqrt{2}}{2} d_2^\pm & [H_2, d_2^\pm] &= \mp \frac{\sqrt{6}}{2} d_2^\pm \\ [H_1, g_3^\pm] &= 0 & [H_2, g_3^\pm] &= \mp \frac{\sqrt{6}}{3} g_3^\pm & [H_1, d_3^\pm] &= \pm \sqrt{2} d_3^\pm & [H_2, d_3^\pm] &= 0 \end{aligned}$$

Therefore, we have found out that the d_k^\pm generators correspond to the external \mathbf{a}_2 in the root diagram of \mathbf{g}_2 , whereas the g_k^\pm generators correspond to the internal hexagon (3 and $\bar{3}$ of \mathbf{a}_2).

The remaining non-vanishing commutation relations are:

$$\begin{aligned} [d_k^\pm, d_{k+1}^\pm] &= \pm d_{k+2}^\mp \\ [d_1^+, d_1^-] &= -\frac{1}{2}(\sqrt{2}H_1 - \sqrt{6}H_2) & [d_2^+, d_2^-] &= -\frac{1}{2}(\sqrt{2}H_1 + \sqrt{6}H_2) & [d_3^+, d_3^-] &= \sqrt{2}H_1 \\ [d_k^\pm, g_{k+1}^\mp] &= \mp g_{k+2}^\mp & [d_{k+1}^\pm, g_k^\pm] &= \pm g_{k+2}^\pm \\ [g_k^\pm, g_{k+1}^\mp] &= \mp 3d_{k+2}^\pm & [g_k^\pm, g_{k+1}^\pm] &= 2g_{k+2}^\mp \\ [g_1^+, g_1^-] &= -\frac{1}{2}(3\sqrt{2}H_1 + \sqrt{6}H_2) & [g_2^+, g_2^-] &= \frac{1}{2}(3\sqrt{2}H_1 - \sqrt{6}H_2) & [g_3^+, g_3^-] &= \sqrt{6}H_2 \end{aligned}$$

we now introduce the following complex algebra of 4×4 Zorn-type matrices:

$$\begin{bmatrix} a & A^+ \\ A^- & t(a) \end{bmatrix} \quad (4.1)$$

where a is a 3×3 complex matrix, $A^+, A^- \in \mathbf{C}^3$, viewed as column and row vectors respectively and $t(a)$ denotes the trace of a .

The product of two such matrices is defined by:

$$\begin{aligned} &\begin{bmatrix} a & A^+ \\ A^- & t(a) \end{bmatrix} \begin{bmatrix} b & B^+ \\ B^- & t(b) \end{bmatrix} \\ &= \begin{bmatrix} ab + A^+ \circ B^- & aB^+ + A^- \wedge B^- \\ A^-b + A^+ \wedge B^+ & t(a)t(b) + t(B^+ \circ A^-) \end{bmatrix}, \end{aligned} \quad (4.2)$$

where

$$A^+ \circ B^- = t(A^+ B^-)I - t(I)A^+ B^- \quad (4.3)$$

(with standard matrix products of row and column vectors and with I denoting the 3×3 identity matrix); $A \wedge B$ is the standard vector product of A and B . Notice that $t(X^+ \circ Y^-) = 0$, hence we have an algebra.

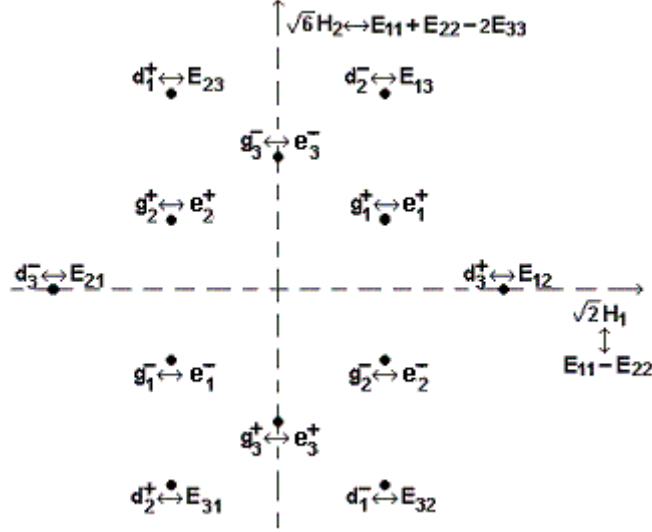


Figure 3: Diagram of roots of \mathfrak{g}_2 with corresponding generators and *matrix-like* elements

In particular, we get a sub-algebra by imposing the a matrices to be traceless. We use this algebra in order to define the following adjoint representation ϱ of the Lie algebra \mathfrak{g}_2 :

$$\begin{bmatrix} a & A^+ \\ A^- & 0 \end{bmatrix} \quad (4.4)$$

where $a \in \mathfrak{a}_2$, A^+ , $A^- \in \mathbf{C}^3$, viewed as column and row vector respectively.

Indeed, the commutator of two such matrices, using (4.2), can be computed to read :

$$\begin{aligned} & \left[\begin{bmatrix} a & A^+ \\ A^- & 0 \end{bmatrix}, \begin{bmatrix} b & B^+ \\ B^- & 0 \end{bmatrix} \right] \\ &= \begin{bmatrix} [a, b] + A^+ \circ B^- - B^+ \circ A^- & aB^+ - bA^+ + 2A^- \wedge B^- \\ A^-b - B^-a + 2A^+ \wedge B^+ & 0 \end{bmatrix}, \end{aligned} \quad (4.5)$$

and therefore one is led to the following identifications of the \mathfrak{g}_2 generators shown above:

$$\begin{aligned} \varrho(d_k^\pm) &= E_{k\pm 1 \ k\pm 2} \pmod{3}, \quad k = 1, 2, 3 \\ \varrho(\sqrt{2}H_1) &= E_{11} - E_{22} & \varrho(\sqrt{6}H_2) &= E_{11} + E_{22} - 2E_{33} \\ \varrho(g_k^+) &= E_{k4} := \mathbf{e}_k^+ & \varrho(g_k^-) &= E_{4k} := \mathbf{e}_k^- \quad , \quad k = 1, 2, 3 \end{aligned} \quad (4.6)$$

where E_{ij} denotes the matrix with all zero elements except a 1 in the $\{ij\}$ position: $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ and \mathbf{e}_k^+ are the standard basis vectors of \mathbf{C}^3 (\mathbf{e}_k^- denote their transpose).

On the other hand, a direct calculation shows that:

$$\varrho([X, Y]) = [\varrho(X), \varrho(Y)] \quad X, Y \in \mathfrak{g}_2 \quad (4.7)$$

thus proving that ϱ (which is obviously linear) is indeed a representation.

It is useful to extend this correspondence to the roots of \mathfrak{g}_2 , obtaining the pictorial view of the diagram in figure 3.

For future use, we here explicitly associate a matrix in the form (4.4) to the derivation $D_{c,d}$ for $c, d \in \mathfrak{C}$. Let us define $e_{jk} := D_{\varepsilon_k^-, \varepsilon_j^+}$ (notice the switch of indices). A straightforward calculation shows that, for $c, d \in \mathfrak{C}$, $c = \alpha_0^\pm \rho^\pm + v_k^\pm \varepsilon_k^\pm$, $d = \beta_0^\pm \rho^\pm + w_k^\pm \varepsilon_k^\pm$, it holds that:

$$\begin{aligned} D_{c,d} &= \frac{1}{3} \left((\alpha_0^\pm - \alpha_0^\mp) w_k^\pm - (\beta_0^\pm - \beta_0^\mp) v_k^\pm - (v^\mp \wedge w^\mp)_k \right) g_k^\pm \\ &\quad + (v_k^- w_j^+ - v_j^+ w_k^-) e_{jk} \end{aligned} \quad (4.8)$$

Notice that $\varrho(e_{ij}) = E_{ij}$ for $i \neq j = 1, 2, 3$, whereas

$$\begin{aligned}\varrho(e_{11}) &= \varrho\left(\frac{\sqrt{2}}{2}H_1 + \frac{\sqrt{6}}{6}H_2\right) = \frac{1}{3}(2E_{11} - E_{22} - E_{33}) \\ \varrho(e_{22}) &= \varrho\left(-\frac{\sqrt{2}}{2}H_1 + \frac{\sqrt{6}}{6}H_2\right) = \frac{1}{3}(-E_{11} - E_{22} + 2E_{33}) \\ \varrho(e_{33}) &= \varrho\left(-\frac{\sqrt{6}}{3}H_2\right) = \frac{1}{3}(-E_{11} - E_{22} + 2E_{33})\end{aligned}$$

We thus obtain from (4.8):

$$\begin{aligned}\varrho(D_{c,d}) &= \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{pmatrix}, \quad \text{where} \\ D_{11} &= -\frac{1}{3}(v_i^- w_i^+ - v_i^+ w_i^-)I + (v_j^- w_i^+ - v_i^+ w_j^-)E_{ij} \\ D_{12} &= \frac{1}{3}((\alpha_0^+ - \alpha_0^-)w^+ - (\beta_0^+ - \beta_0^-)v^+ - (v^- \wedge w^-)) \\ D_{21} &= \frac{1}{3}((\alpha_0^- - \alpha_0^+)w^- - (\beta_0^- - \beta_0^+)v^- - (v^+ \wedge w^+))\end{aligned} \quad (4.9)$$

We introduce the following action of $\varrho(\mathbf{g}_2)$ on the octonions represented by Zorn matrices:

$$\begin{aligned}&\left[\begin{bmatrix} a & A^+ \\ A^- & 0 \end{bmatrix}, \begin{bmatrix} \alpha_0^+ & v^+ \\ v^- & \alpha_0^- \end{bmatrix} \right] \\ &= \begin{bmatrix} -v^- A^+ + A^- v^+ & av^+ + (\alpha_0^- - \alpha_0^+)A^+ - A^- \wedge v^- \\ -v^- a - (\alpha_0^- - \alpha_0^+)A^- - A^+ \wedge v^+ & v^- A^+ - A^- v^+ \end{bmatrix}\end{aligned} \quad (4.10)$$

We see that $\varrho(\mathbf{g}_2)$ acts non-trivially on traceless octonions, hence we can write $\alpha_0^+ = -\alpha_0^-$ to get a 'matrix-like' expression of the 7-dimensional (fundamental) representation $\mathbf{7}$ of \mathbf{g}_2 .

A direct calculation confirms that the action (4.10) corresponds to the action of the \mathbf{g}_2 generators on the octonions shown above.

It can also be shown that the action (4.10) is indeed a derivation of the octonions, confirming that $\text{Der}(\mathfrak{C}) = \mathbf{g}_2$. The only ingredients needed for the proof are identities from elementary 3-dimensional geometry, like $(A \wedge B) \cdot C = (C \wedge A) \cdot B$ or $(A \wedge B) \wedge C = (A \cdot B)C - (B \cdot C)A$, plus the following identity for a 3×3 traceless matrix a :

$$\sum_j a_{ij} \epsilon_{jkl} + a_{kj} \epsilon_{ijl} + a_{lj} \epsilon_{ikj} = 0$$

5 $\mathbf{n} = 1$: Matrix representation of \mathbf{f}_4

We introduce in this section the representation ϱ of \mathbf{f}_4 in the form of a matrix. For $\mathbf{f} \in \mathbf{f}_4$:

$$\varrho(\mathbf{f}) = \begin{pmatrix} a \otimes I + I \otimes a_1 & \mathbf{x}^+ \\ \mathbf{x}^- & -I \otimes a_1^T \end{pmatrix}, \quad (5.1)$$

where $a \in \mathbf{a}_2^{(1)}$, $a_1 \in \mathbf{a}_2^{(2)}$ (the superscripts being merely used to distinguish the two copies of \mathbf{a}_2) a_1^T is the transpose of a_1 , I is the 3×3 identity matrix, $\mathbf{x}^+ \in \mathbf{C}^3 \otimes \mathbf{J}_3^1$, $\mathbf{x}^- \in \mathbf{C}^3 \otimes \bar{\mathbf{J}}_3^1$:

$$\mathbf{x}^+ := \begin{pmatrix} x_1^+ \\ x_2^+ \\ x_3^+ \end{pmatrix} \quad \mathbf{x}^- := (x_1^-, x_2^-, x_3^-), \quad x_i^+ \in \mathbf{J}_3^1 \quad x_i^- \in \bar{\mathbf{J}}_3^1, \quad i = 1, 2, 3$$

The commutator is set to be:

$$\begin{aligned} & \left[\begin{pmatrix} a \otimes I + I \otimes a_1 & \mathbf{x}^+ \\ \mathbf{x}^- & -I \otimes a_1^T \end{pmatrix}, \begin{pmatrix} b \otimes I + I \otimes b_1 & \mathbf{y}^+ \\ \mathbf{y}^- & -I \otimes b_1^T \end{pmatrix} \right] \\ & := \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \end{aligned} \quad (5.2)$$

where:

$$\begin{aligned} C_{11} &= [a, b] \otimes I + I \otimes [a_1, b_1] + \mathbf{x}^+ \diamond \mathbf{y}^- - \mathbf{y}^+ \diamond \mathbf{x}^- \\ C_{12} &= (a \otimes I) \mathbf{y}^+ - (b \otimes I) \mathbf{x}^+ + (I \otimes a_1) \mathbf{y}^+ + \mathbf{y}^+ (I \otimes a_1^T) \\ & \quad - (I \otimes b_1) \mathbf{x}^+ - \mathbf{x}^+ (I \otimes b_1^T) + \mathbf{x}^- \times \mathbf{y}^- \\ C_{21} &= -\mathbf{y}^- (a \otimes I) + \mathbf{x}^- (b \otimes I) - (I \otimes a_1^T) \mathbf{y}^- - \mathbf{y}^- (I \otimes a_1) \\ & \quad + (I \otimes b_1^T) \mathbf{x}^- + \mathbf{x}^- (I \otimes b_1) + \mathbf{x}^+ \times \mathbf{y}^+ \\ C_{22} &= I \otimes [a_1^T, b_1^T] + \mathbf{x}^- \bullet \mathbf{y}^+ - \mathbf{y}^- \bullet \mathbf{x}^+ \end{aligned} \quad (5.3)$$

with the following definitions :

$$\begin{aligned} \mathbf{x}^+ \diamond \mathbf{y}^- &:= \left(\frac{1}{3} t(x_i^+, y_i^-) I - t(x_i^+, y_j^-) E_{ij} \right) \otimes I + \\ & \quad I \otimes \left(\frac{1}{3} t(x_i^+, y_i^-) I - x_i^+ y_i^- \right) \\ \mathbf{x}^- \bullet \mathbf{y}^+ &:= I \otimes \left(\frac{1}{3} t(x_i^-, y_i^+) I - x_i^- y_i^+ \right) \\ (\mathbf{x}^\pm \times \mathbf{y}^\pm)_i &:= \epsilon_{ijk} [x_j^\pm y_k^\pm + y_k^\pm x_j^\pm - x_j^\pm t(y_k^\pm) - y_k^\pm t(x_j^\pm) \\ & \quad - (t(x_j^\pm, y_k^\pm) - t(x_j^\pm) t(y_k^\pm)) I] \\ &:= \epsilon_{ijk} (x_j^\pm \# y_k^\pm) \end{aligned} \quad (5.4)$$

Notice that:

1. $x \in \mathbf{J}_3^1$ is a symmetric complex matrix;
2. writing $\mathbf{x}^+ \diamond \mathbf{y}^- := c \otimes I + I \otimes c_1$ we have that both c and c_1 are traceless hence $c, c_1 \in \mathbf{a}_2$, and indeed they have 8 (complex) parameters, and $\mathbf{y}^- \bullet \mathbf{x}^+ = I \otimes c_1^T$;
3. terms like $(I \otimes a_1) \mathbf{y}^+ + \mathbf{y}^+ (I \otimes a_1^T)$ are in $\mathbf{C}^3 \otimes \mathbf{J}_3^1$, namely they are matrix valued vectors with symmetric matrix elements;
4. the *sharp* product $\#$ of \mathbf{J}_3^1 matrices appearing in $\mathbf{x}^\pm \times \mathbf{y}^\pm$ is the fundamental product in the theory of Jordan Algebras, introduced in section 2.

In order to prove that ϱ is a representation of the Lie algebra \mathbf{f}_4 we make a comparison with Tits' construction of the fourth row of the magic square, [57] [58]. If \mathbf{J}_0 denotes the traceless elements of \mathbf{J} , \mathfrak{C}_0 the traceless octonions (the trace being defined by $t(a) := a + \bar{a} \in \mathbf{C}$, for $a \in \mathfrak{C}$ where the bar denotes the octonion conjugation - that does not affect the field \mathbf{C} -), it holds that :

$$\mathbf{f}_4 = \text{Der}(\mathfrak{C}) \oplus (\mathfrak{C}_0 \otimes \mathbf{J}_0) \oplus \text{Der}(\mathbf{J}) \quad (5.5)$$

with commutation rules, for $D \in \text{Der}(\mathfrak{C}) = \mathfrak{g}_2$, $c, d \in \mathfrak{C}_0$, $x, y \in \mathbf{J}_0$, $E \in \text{Der}(\mathbf{J})$, given by:

$$\begin{aligned}
[\text{Der}(\mathfrak{C}), \text{Der}(\mathfrak{C})] &= \text{Der}(\mathfrak{C}) \\
[\text{Der}(\mathbf{J}), \text{Der}(\mathbf{J})] &= \text{Der}(\mathbf{J}) \\
[\text{Der}(\mathfrak{C}), \text{Der}(\mathbf{J})] &= 0 \\
[D, c \otimes x] &= D(c) \otimes x \\
[E, c \otimes x] &= c \otimes E(x) \\
[c \otimes x, d \otimes y] &= t(xy)D_{c,d} + 2(c * d) \otimes (x * y) + \frac{1}{2}t(cd)[x, y]
\end{aligned} \tag{5.6}$$

where , $\mathfrak{C}_0 \ni c * d = cd - \frac{1}{2}t(cd)$, $\mathbf{J}_0 \ni x * y = \frac{1}{2}(xy + yx) - \frac{1}{3}t(xy)I$, $\mathbf{J} = \mathbf{J}_3^1$.

The derivations of \mathbf{J} are inner: $\text{Der}(\mathbf{J}) = [\tilde{L}(\mathbf{J}), L(\mathbf{J})]$ where L stands for the left (or right) multiplication with respect to the Jordan product: $L_x y = \frac{1}{2}(xy + yx)$. In the case under consideration, the product $x, y \rightarrow xy$ is associative and $[L_x, L_y]z = \frac{1}{4}[[x, y], z]$. Since $[x, y]$ is antisymmetric, then $\text{Der}(\mathbf{J}) = \mathfrak{so}(3)_{\mathbf{C}} \equiv \mathfrak{a}_1$.

We can thus put forward the following correspondence:

$$\begin{aligned}
\varrho(D) &= \begin{pmatrix} a \otimes I & \frac{1}{3}tr(\mathbf{x}^+) \otimes I \\ \frac{1}{3}tr(\mathbf{x}^-) \otimes I & 0 \end{pmatrix} \\
\varrho(E) &= \begin{pmatrix} I \otimes a_1^A & 0 \\ 0 & I \otimes a_1^A \end{pmatrix} \\
\varrho(\varepsilon_k^+ \otimes \mathbf{J}_0) &= \begin{pmatrix} 0 & \mathbf{x}_k^+ - \frac{1}{3}tr(\mathbf{x}_k^+) \otimes I \\ 0 & 0 \end{pmatrix} \\
\varrho(\varepsilon_k^- \otimes \mathbf{J}_0) &= \begin{pmatrix} 0 & 0 \\ \mathbf{x}_k^- - \frac{1}{3}tr(\mathbf{x}_k^-) \otimes I & 0 \end{pmatrix} \\
\varrho((\rho^+ - \rho^-) \otimes \mathbf{J}_0) &= \begin{pmatrix} I \otimes a_1^S & 0 \\ 0 & -I \otimes a_1^S \end{pmatrix},
\end{aligned} \tag{5.7}$$

where a_1^A and a_1^S are the antisymmetric and symmetric parts of a_1 , and

$$tr(\mathbf{x}^+) := \begin{pmatrix} t(x_1^+) \\ t(x_2^+) \\ t(x_3^+) \end{pmatrix}, \quad tr(\mathbf{x}^-) = (t(x_1^-), t(x_2^-), t(x_3^-))$$

with \mathbf{x}_k^\pm denoting a matrix-valued vector whose k -th component is the only non-vanishing one.

5.1 Comparison with Tits' construction

It is here worth commenting that there is some apparent difference between the way we write Tits' construction, (5.6), and the way it is written in the mathematical literature; see for instance [59], page93.

Firstly, we have the operators acting from the left, contrary to the action from the right often used by mathematicians. This implies that the third and fourth commutators in (5.6) are

written in the reverse order. Moreover, the last commutator of (5.6) is instead written in [59] (using the superscript $^\top$ in order to distinguish it from ours) as follows :

$$[c \otimes x, d \otimes y]^\top = \frac{1}{12}t(xy)D_{c,d}^\top + ((c * d) \otimes (x * y))^\top + \frac{1}{2}t(cd)[L_x, L_y]. \quad (5.8)$$

Furthermore, we observe that we have defined the derivation $D_{a,b} = \frac{1}{3}D_{a,b}^\top$ (up to a sign due to left *versus* right action). Because of this and the fact that $[L_x, L_y]z = \frac{1}{4}[[x, y], z]$, we have 4 times the first and third terms in (5.8), and 2 times the middle one. However, these factors can be reabsorbed by changing $\varrho(c \times x) \rightarrow \frac{1}{2}\varrho(c \times x)$, thus proving the equivalence of the two ways of writing all the commutation relations.

5.2 ϱ as a representation of \mathfrak{f}_4

By exploiting the correspondence (5.7), we now prove in six steps that the commutators (5.3) satisfy (5.6), thus proving the following

Theorem : ϱ realizes the adjoint representation of \mathfrak{f}_4 .

Proof : 1) $[Der(\mathfrak{C}), Der(\mathfrak{C})] = Der(\mathfrak{C})$

In order to prove this first step, let us denote by A^\pm and B^\pm the \mathbf{C}^3 vectors $\frac{1}{3}tr(\mathbf{x}^\pm)$ and $\frac{1}{3}tr(\mathbf{y}^\pm)$ respectively. Then, we have to compute:

$$\left[\begin{pmatrix} a \otimes I & A^+ \otimes I \\ A^- \otimes I & 0 \end{pmatrix}, \begin{pmatrix} b \otimes I & B^+ \otimes I \\ B^- \otimes I & 0 \end{pmatrix} \right] := \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \quad (5.9)$$

Let us calculate some terms separately:

$$A^+ \otimes I \diamond B^- \otimes I = (A_i^+ B_i^- I - 3A_i^+ B_j^- E_{ij}) \otimes I + I \otimes (\frac{1}{3}A_i^+ B_i^- t(I)I - A_i^+ B_i^- I);$$

the first bracket on the right-hand side is $A^+ \circ B^-$, as defined in (4.3), whereas the second one vanishes.

$$(A^+ \otimes I) \times (B^+ \otimes I)_i = \epsilon_{ijk} A_j^+ B_k^+ (2I - 2t(I)I - (t(I) - t(I)^2)I) = 2(A^+ \wedge B^+)_i I;$$

similarly with A^- and B^- , hence

$$(A^\pm \otimes I) \times (B^\pm \otimes I) = 2(A^\pm \wedge B^\pm) \otimes I.$$

Therefore, we obtain

$$\begin{aligned} C_{11} &= ([a, b] + A^+ \circ B^- - B^+ \circ A^-) \otimes I \\ C_{12} &= (aB^+ - bA^+ + 2A^- \wedge B^-) \otimes I \\ C_{21} &= (-B^- a + A^- b + 2A^+ \wedge B^+) \otimes I \\ C_{22} &= 0, \end{aligned} \quad (5.10)$$

which make the commutation relations (5.9) correspond to those of \mathfrak{g}_2 introduced in (4.5).

$$2), 3) [Der(\mathbf{J}), Der(\mathbf{J})] = Der(\mathbf{J}), [Der(\mathfrak{C}), Der(\mathbf{J})] = 0$$

One can prove this in a straightforward way, *e.g.* by explicit computation.

$$4) [D, c \otimes x] = D(c) \otimes x$$

In order to prove this, let us write $c = \alpha(\rho^+ - \rho^-) + v_k^\pm \varepsilon_k^\pm \in \mathfrak{C}_0$ (summed over \pm and k), and let us consider $V^\pm \in \mathbf{C}^3$ with components v_k^\pm . Then, we have to calculate:

$$\left[\begin{pmatrix} a \otimes I & A^+ \otimes I \\ A^- \otimes I & 0 \end{pmatrix}, \begin{pmatrix} \alpha I \otimes x & V^+ \otimes x \\ V^- \otimes x & -\alpha I \otimes x \end{pmatrix} \right] := \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \quad (5.11)$$

Let us calculate some terms separately:

$$\begin{aligned} A^+ \otimes I \diamond V^- \otimes x &= \left(\frac{1}{3} A_i^+ v_i^- t(x) I - A_i^+ v_j^- t(x) E_{ij} \right) \otimes I \\ &\quad + I \otimes \left(\frac{1}{3} A_i^+ v_i^- t(x) I - A_i^+ v_i^- x \right) \\ &= -I \otimes A_i^+ v_i^- x, \end{aligned}$$

since $t(x) = 0$ by hypothesis.

$$(A^+ \otimes I) \times (V^+ \otimes I)_i = \epsilon_{ijk} A_j^+ v_k^+ (2x - xt(I)) = -(A^+ \wedge v^+)_i x,$$

once again because $t(x) = 0$. Similarly with A^- and v^- , hence

$$(A^\pm \otimes I) \times (V^\pm \otimes x) = -(A^\pm \wedge V^\pm) \otimes x$$

Consequently, we obtain

$$\begin{aligned} C_{11} &= I \otimes (-v_i^- A_i^+ + A_i^- v_i^+) x \\ C_{12} &= (aV^+ - 2\alpha A^+ - A^- \wedge B^-) \otimes x \\ C_{21} &= (-V^- a + 2\alpha A^- - A^+ \wedge V^+) \otimes x \\ C_{22} &= I \otimes (v_i^- A_i^+ - A_i^- v_i^+) x, \end{aligned} \quad (5.12)$$

which is the \mathbf{g}_2 action on $c \in \mathfrak{C}_0$ introduced in (4.10) tensored with x .

5) $[E, a \otimes x] = a \otimes E(x)$ (this can be proved by explicit computation).

6) $[c \otimes x, d \otimes y] = t(xy) D_{c,d} + 2(c * d) \otimes (x * y) + \frac{1}{2} t(cd)[x, y]$

Let us use notations analogous to the ones in the proof of 4). Then, we have to compute:

$$\left[\begin{pmatrix} \alpha I \otimes x & V^+ \otimes x \\ V^- \otimes x & -\alpha I \otimes x \end{pmatrix}, \begin{pmatrix} \beta I \otimes y & W^+ \otimes y \\ W^- \otimes y & -\beta I \otimes y \end{pmatrix} \right] := \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \quad (5.13)$$

Let us calculate some terms separately:

$$\begin{aligned} (V^+ \otimes x) \diamond (W^- \otimes y) &= \left(\frac{1}{3} v_i^+ w_i^- t(xy) I - v_i^+ w_j^- t(xy) E_{ij} \right) \otimes I \\ &\quad + I \otimes \left(\frac{1}{3} v_i^+ w_i^- t(xy) I - v_i^+ w_i^- xy \right) \end{aligned}$$

$$(V^+ \otimes x) \times (W^+ \otimes y)_i = \epsilon_{ijk} v_j^+ w_k^+ (xy + yx - t(xy) I)$$

being $t(x) = t(y) = 0$. Similarly with A^- and v^- , hence

$$(V^\pm \otimes x) \times (W^\pm \otimes y) = -(V^\pm \wedge W^\pm) \otimes (xy + yx - t(xy) I)$$

Therefore, one obtains

$$\begin{aligned}
C_{11} &= \left(\frac{1}{3}(v_i^+ w_i^- - w_i^+ v_i^-)t(xy)I - (v_i^+ w_j^- - w_i^+ v_j^-)t(xy)E_{ij} \right) \otimes I \\
&\quad + I \otimes \left(\alpha\beta[x, y] + \frac{1}{3}(v_i^+ w_i^- - w_i^+ v_i^-)t(xy)I - v_i^+ w_i^- xy + w_i^+ v_i^- yx \right) \\
&= \left(\frac{1}{3}(v_i^+ w_i^- - w_i^+ v_i^-)t(xy)I - (v_i^+ w_j^- - w_i^+ v_j^-)t(xy)E_{ij} \right) \otimes I \\
&\quad + I \otimes \left(\left(\alpha\beta - \frac{1}{2}(v_i^- w_i^+ + v_i^+ w_i^-) \right) [x, y] \right. \\
&\quad \left. + (v_i^- w_i^+ - w_i^- v_i^+) \left(\frac{1}{2}(xy + yx) - \frac{1}{3}t(xy)I \right) \right) \\
C_{12} &= (\alpha W^+ - \beta V^+) \otimes (xy + yx) + (V^- \wedge W^-) \otimes (xy + yx - t(xy)I) \\
&= 2(\alpha W^+ - \beta V^+ + V^- \wedge W^-) \otimes \left(\frac{1}{2}(xy + yx) - \frac{1}{3}t(xy)I \right) \\
&\quad + (2\alpha W^+ - 2\beta V^+ - V^- \wedge W^-) \otimes \frac{1}{3}t(xy)I
\end{aligned}$$

with similar results for C_{21} and C_{22} .

Finally, for $c, d \in \mathfrak{C}_0$, $c = \alpha(\rho^+ - \rho^-) + v_k^\pm \varepsilon_k^\pm$, $d = \beta(\rho^+ - \rho^-) + w_k^\pm \varepsilon_k^\pm$ (summed over \pm and k), $V^\pm, W^\pm \in \mathbf{C}^3$ with components v_k^\pm, v_k^\pm , it can be computed that :

$$\begin{aligned}
c * d &= \frac{1}{2}(v_k^- w_k^+ - v_k^+ w_k^-)(\rho^+ - \rho^-) + (\pm \alpha w_k^\pm \mp \beta v_k^\pm + (V^\mp \wedge W^\mp)_k) \varepsilon_k^\pm \\
\frac{1}{2}t(cd) &= \alpha\beta - \frac{1}{2}(v_k^- w_k^+ - v_k^+ w_k^-)
\end{aligned} \tag{5.14}$$

From (5.14) and (4.9) we obtain indeed the proof of 6).

This completes the proof that ϱ (5.1) is the adjoint representation of \mathbf{f}_4 . ■

Notice that (5.1) reproduces the well known branching rule of the adjoint of \mathbf{f}_4 with respect to its maximal and non-symmetric subalgebra $\mathbf{a}_2^{(1)} \oplus \mathbf{a}_2^{(2)}$:

$$\mathbf{52} = (\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{8}) + (\mathbf{3}, \overline{\mathbf{6}}) + (\overline{\mathbf{3}}, \mathbf{6}). \tag{5.15}$$

It is here worth anticipating that in section 8 we prove the Jacobi identity in the more general case of \mathbf{e}_7 , which includes in an obvious manner this case of \mathbf{f}_4 . The validity of the Jacobi identity, together with the fact that the representation ϱ fulfills the root diagram of \mathbf{f}_4 (the proof is straightforward, and it can also be considered as a particular case of the proof given at the end of section 6 for \mathbf{e}_6) proves in an alternative way that ϱ is indeed a representation of \mathbf{f}_4 .

6 $\mathfrak{n} = 2$: Matrix representation of \mathbf{e}_6

We present in this section the representation ϱ of \mathbf{e}_6 in the form of a matrix. We have to complexify the Jordan structure with respect to \mathbf{f}_4 . We introduce the imaginary unit \mathbf{u}_1 - leaving \mathbf{i} as the imaginary unit of the base field. In particular, \mathbf{J}_3^2 is Hermitian with respect to the \mathbf{u}_1 -conjugation, and we are going to denote such an Hermitian conjugation with the symbol \dagger throughout.

In a similar fashion to (5.1), for $\mathbf{f} \in \mathbf{e}_6$, we thus write:

$$\varrho(\mathbf{f}) = \begin{pmatrix} a \otimes I + I \otimes a_1 & \mathbf{x}^+ \\ \mathbf{x}^- & -I \otimes a_1^\dagger \end{pmatrix} \tag{6.1}$$

where $a \in \mathbf{a}_2^{\mathbf{f}}$, $a_1 \in \mathbf{a}_2^{(1)} \oplus \mathbf{u}_1 \mathbf{a}_2^{(2)}$, a_1^\dagger is the Hermitian conjugate of a_1 (with respect to \mathbf{u}_1), I is the 3×3 identity matrix, $\mathbf{x}^+ \in \mathbf{C}^3 \otimes \mathbf{J}_3^2$, $\mathbf{x}^- \in \mathbf{C}^3 \otimes \overline{\mathbf{J}}_3^2$:

$$\mathbf{x}^+ = \begin{pmatrix} x_1^+ \\ x_2^+ \\ x_3^+ \end{pmatrix} \quad \mathbf{x}^- = (x_1^-, x_2^-, x_3^-), \quad x_i^+ \in \mathbf{J}_3^2, \quad x_i^- \in \overline{\mathbf{J}}_3^2, \quad i = 1, 2, 3$$

The commutator of two such matrices is the same as for \mathbf{f}_4 , with \dagger instead of T (cfr. (5.2)):

$$\begin{aligned} & \left[\begin{pmatrix} a \otimes I + I \otimes a_1 & \mathbf{x}^+ \\ \mathbf{x}^- & -I \otimes a_1^\dagger \end{pmatrix}, \begin{pmatrix} b \otimes I + I \otimes b_1 & \mathbf{y}^+ \\ \mathbf{y}^- & -I \otimes b_1^\dagger \end{pmatrix} \right] \\ & := \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \end{aligned} \quad (6.2)$$

where:

$$\begin{aligned} C_{11} &= [a, b] \otimes I + I \otimes [a_1, b_1] + \mathbf{x}^+ \diamond \mathbf{y}^- - \mathbf{y}^+ \diamond \mathbf{x}^- \\ C_{12} &= (a \otimes I) \mathbf{y}^+ - (b \otimes I) \mathbf{x}^+ + (I \otimes a_1) \mathbf{y}^+ + \mathbf{y}^+ (I \otimes a_1^\dagger) \\ & \quad - (I \otimes b_1) \mathbf{x}^+ - \mathbf{x}^+ (I \otimes b_1^\dagger) + \mathbf{x}^- \times \mathbf{y}^- \\ C_{21} &= -\mathbf{y}^- (a \otimes I) + \mathbf{x}^- (b \otimes I) - (I \otimes a_1^\dagger) \mathbf{y}^- - \mathbf{y}^- (I \otimes a_1) \\ & \quad + (I \otimes b_1^\dagger) \mathbf{x}^- + \mathbf{x}^- (I \otimes b_1) + \mathbf{x}^+ \times \mathbf{y}^+ \\ C_{22} &= I \otimes [a_1^\dagger, b_1^\dagger] + \mathbf{x}^- \bullet \mathbf{y}^+ - \mathbf{y}^- \bullet \mathbf{x}^+ \end{aligned} \quad (6.3)$$

with products defined as in (5.4).

Notice that:

1. $x \in \mathbf{J}_3^2$ is a Hermitian matrix (with respect to \mathbf{u}_1) over the complex field (with imaginary unit \mathbf{i});
2. by writing $a_1 \in \mathbf{a}_2^{(1)} \oplus \mathbf{u}_1 \mathbf{a}_2^{(2)}$ we state that a_1 is the sum of a traceless skew-Hermitian matrix and a traceless hermitian matrix (namely a matrix in \mathbf{J}_0 , with $\mathbf{J} = \mathbf{J}_3^2$), hence $a_1 \in \mathbf{sl}(\mathbf{3}, \mathbf{C})$ is a generic 3×3 traceless matrix over $\mathbf{C} \otimes \mathbf{C}$;
3. writing $\mathbf{x}^+ \diamond \mathbf{y}^- := c \otimes I + I \otimes c_1$ we have that both c and c_1 are traceless, $c \in \mathbf{a}_2$ and $c_1 \in \mathbf{sl}(\mathbf{3}, \mathbf{C})$, and $\mathbf{y}^- \bullet \mathbf{x}^+ = I \otimes c_1^\dagger$; if $x, y \in \mathbf{J}_3^2$, then $\mathbf{C} \ni t(x, y) = t(xy)$, and c_1 has indeed 16 (complex) parameters. It is here worth anticipating that this will not be the case for \mathbf{J}_3^4 and \mathbf{J}_3^8 , as we shall stress in the next sections on \mathbf{e}_7 and \mathbf{e}_8 ;
4. terms like $(I \otimes a_1) \mathbf{y}^+ + \mathbf{y}^+ (I \otimes a_1^\dagger)$ are in $\mathbf{C}^3 \otimes \mathbf{J}_3^2$, namely they are matrix valued vectors with Hermitian matrix elements;
5. the correspondence between matrix elements in (6.1) and Tits' construction is similar to the one shown in (5.7) and is omitted here;
6. the Jacobi identity can be demonstrated as a particular case of the proof for \mathbf{e}_7 , shown in section 8. The validity of the Jacobi identity, together with the fact that the representation ϱ fulfills the root diagram of \mathbf{e}_6 , as we show next, prove that ϱ (6.1) is the adjoint representation of \mathbf{e}_6 . ■

As regards the counting of parameters, we refer to our comment in the introduction about the use of \mathbf{C} as base field.

We end this section with the correspondence between the roots of \mathbf{e}_6 and the matrix elements in (6.1).

The roots of \mathbf{e}_6 can be written in terms of an orthonormal basis $\{k_i \mid i = 1, \dots, 6\}$ as, [36]:

\mathbf{e}_6	72 roots
$\pm k_i \pm k_j \quad i \neq j = 1, \dots, 5$	$4 \times \binom{5}{2} = 40$
$\frac{1}{2}(\pm k_1 \pm k_2 \pm k_3 \pm k_4 \pm k_5 \pm \sqrt{3}k_6)^*$	$2^5 = 32$
* [odd number of + signs]	

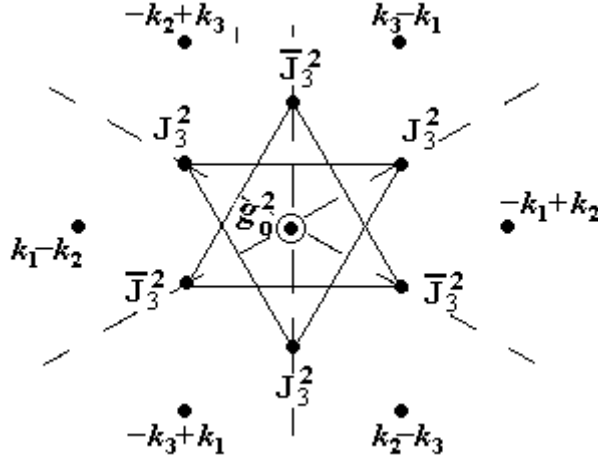


Figure 4: Roots of \mathfrak{e}_6 projected on the plane of \mathfrak{g}_2

We refer to Figure 4 and write the roots associated with the highest weight \mathbf{J}_3^2 as:

$$\begin{aligned}
& -k_1 \pm k_4, \quad -k_1 \pm k_5, \quad k_2 + k_3 \\
& \frac{1}{2}(-k_1 + k_2 + k_3 + k_4 - k_5 - \sqrt{3}k_6) \\
& \frac{1}{2}(-k_1 + k_2 + k_3 + k_4 + k_5 + \sqrt{3}k_6) \\
& \frac{1}{2}(-k_1 + k_2 + k_3 - k_4 + k_5 - \sqrt{3}k_6) \\
& \frac{1}{2}(-k_1 + k_2 + k_3 - k_4 - k_5 + \sqrt{3}k_6)
\end{aligned} \tag{6.4}$$

The other \mathbf{J}_3^2 's correspond to a cyclic permutation of k_1, k_2, k_3 , and each $\bar{\mathbf{J}}_3^2$ in a Jordan pair $(\mathbf{J}_3^2, \bar{\mathbf{J}}_3^2)$ corresponds to the roots of a \mathbf{J}_3^2 with opposite signs.

The subalgebra $\mathfrak{g}_0^2 \simeq \mathfrak{a}_2 \oplus \mathfrak{a}_2$ has roots:

$$\begin{aligned}
& \pm(k_4 + k_5) \\
& \pm\frac{1}{2}(k_1 + k_2 + k_3 - k_4 - k_5 - \sqrt{3}k_6) \\
& \pm\frac{1}{2}(k_1 + k_2 + k_3 + k_4 + k_5 - \sqrt{3}k_6),
\end{aligned} \tag{6.5}$$

and

$$\begin{aligned}
& \pm(k_4 - k_5) \\
& \pm\frac{1}{2}(k_1 + k_2 + k_3 - k_4 + k_5 + \sqrt{3}k_6) \\
& \pm\frac{1}{2}(k_1 + k_2 + k_3 + k_4 - k_5 + \sqrt{3}k_6).
\end{aligned} \tag{6.6}$$

Furthermore, the roots of \mathfrak{a}_2^f relate in the standard way to the matrix elements of $a \otimes I$ in (6.1). The roots of each Jordan Pair project on the plane of \mathfrak{a}_2^f according to Figure 4, as it can be easily checked. Therefore, we are only left with the roots corresponding to $\mathfrak{sl}(3, \mathbb{C})$ and to each matrix element of a \mathbf{J}_3^2 , say the highest weight one. The rest of the correspondence will readily follow.

The algebras $\mathfrak{a}_2^{(1)}$ and $\mathfrak{a}_2^{(2)}$ are related to Tits' construction. Now, we twist them in the following way: we denote by $\rho^\pm := \frac{1}{2}(1 \pm \mathbf{i}\mathbf{u}_1)$ and introduce $\mathfrak{a}_2^\pm = \rho^\pm \mathfrak{sl}(3, \mathbb{C})$. Then, it follows that $(\rho^\pm)^2 = \rho^\pm$ and $\rho^\pm \rho^\mp = 0$. If $a \in \mathfrak{sl}(3, \mathbb{C})$ then $a = a^+ + a^-$, $a^\pm = \rho^\pm a$ and, if we write $a = a_r + \mathbf{u}_1 a_i$ (where a_r and a_i are the self-conjugate parts of a with respect to \mathbf{u}_1), one can easily check that $a^\pm = (a_r \mp \mathbf{i}a_i)\rho^\pm$. Moreover, for $a, b \in \mathfrak{sl}(3, \mathbb{C})$ then $[a, b] = [a^+ + a^-, b^+ + b^-] = [a^+, b^+] + [a^-, b^-]$. Therefore \mathfrak{a}_2^+ and \mathfrak{a}_2^- are both isomorphic to \mathfrak{a}_2 and $\mathfrak{sl}(3, \mathbb{C}) \simeq \mathfrak{a}_2^+ \oplus \mathfrak{a}_2^-$.

We now write $x \in \mathbf{J}_3^2 = \alpha_i E_{ii} + a_{i,i+1} E_{i,i+1} + \bar{a}_{i,i+1} E_{i+1,i}$, where the indices run over $1, 2, 3 \bmod(3)$, $\alpha \in \mathbb{C}$ and $a_{ij} \in \mathbb{C} \otimes \mathbb{C}$. Obviously $\alpha_i = \alpha_i(\rho^+ + \rho^-)$ and $a_{ij} = a_{ij}(\rho^+ + \rho^-)$. The

matrix x is therefore in the linear span of the nine generators

$$X_i = E_{ii} , \quad X_{i,i+1}^\pm := \rho^\pm E_{i,i+1} + \rho^\mp E_{i+1,i} \quad (6.7)$$

We fix the Cartan subalgebra of $\mathfrak{a}_2^+ \oplus \mathfrak{a}_2^-$ in the obvious way, by introducing the Cartan generators

$$H_{1,2}^\pm := \rho^\pm H_{1,2} , \quad H_1 = \frac{\sqrt{2}}{2} (E_{11} - E_{22}) , \quad H_2 = \frac{\sqrt{6}}{6} (E_{11} + E_{22} - 2E_{33}) \quad (6.8)$$

We let H_1^+, H_2^+ correspond to the axes along the directions of the unit vectors $\frac{\sqrt{2}}{2}(k_4 + k_5)$, $-\frac{\sqrt{6}}{6}(k_1 + k_2 + k_3 - \sqrt{3}k_6)$ and H_1^-, H_2^- to $\frac{\sqrt{2}}{2}(k_4 - k_5)$, $-\frac{\sqrt{6}}{6}(k_1 + k_2 + k_3 + \sqrt{3}k_6)$ respectively.

Consequently, we are all set to establish the correspondence between the roots and the generators of the highest weight \mathbf{J}_3^2 , by exploiting the commutation rule (6.2). This is shown in Table 2.

Table 2: Roots and $\mathfrak{a}_2^+ \oplus \mathfrak{a}_2^-$ weights of the highest weight \mathbf{J}_3^2				
Root	Generator	\mathfrak{a}_2^+ weights	\mathfrak{a}_2^- weights	
$-k_1 + k_4$	X_1	$\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{6}$		
$\frac{1}{2}(-k_1 + k_2 + k_3 + k_4 - k_5 - \sqrt{3}k_6)$	X_{31}^+	$0, -\frac{\sqrt{6}}{3}$	$\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{6}$	
$-k_1 - k_5$	X_{12}^-	$-\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{6}$		
$\frac{1}{2}(-k_1 + k_2 + k_3 + k_4 + k_5 + \sqrt{3}k_6)$	X_{31}^-	$\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{6}$		
$k_2 + k_3$	X_3	$0, -\frac{\sqrt{6}}{3}$	$0, -\frac{\sqrt{6}}{3}$	
$\frac{1}{2}(-k_1 + k_2 + k_3 - k_4 - k_5 + \sqrt{3}k_6)$	X_{23}^+	$-\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{6}$		
$-k_1 + k_5$	X_{12}^+	$\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{6}$		
$\frac{1}{2}(-k_1 + k_2 + k_3 - k_4 + k_5 - \sqrt{3}k_6)$	X_{23}^-	$0, -\frac{\sqrt{6}}{3}$	$-\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{6}$	
$-k_1 - k_4$	X_2	$-\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{6}$		

We thus reproduce the well known branching rule of the adjoint of \mathfrak{e}_6 with respect to its maximal and non-symmetric subalgebra $\mathfrak{a}_2^f \oplus \mathfrak{a}_2^+ \oplus \mathfrak{a}_2^-$:

$$\mathbf{78} = (\mathbf{8}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{8}) + (\mathbf{3}, \mathbf{3}, \mathbf{3}) + (\overline{\mathbf{3}}, \overline{\mathbf{3}}, \overline{\mathbf{3}}) , \quad (6.9)$$

with the exact correspondence of each single root with a matrix elements of (6.1).

It is intriguing to remark the quantum information meaning of the maximal non-symmetric embedding of $\mathfrak{a}_2^f \oplus \mathfrak{a}_2^+ \oplus \mathfrak{a}_2^-$ into \mathfrak{e}_6 has been investigated in [61], within the context of the so-called “black hole - qubit correspondence” [27].

7 $\mathfrak{n} = 4$: Matrix representation of \mathfrak{e}_7

In the present section, we briefly mention how the results of the previous sections can be extended to the case of \mathfrak{e}_7 . Nothing different really occurs, as of course the Jordan algebras involved are of the type \mathbf{J}_3^4 , whose elements associate with respect to the standard product of matrices.

For $\mathfrak{f} \in \mathfrak{e}_7$, we write:

$$\varrho(\mathfrak{f}) = \begin{pmatrix} a \otimes I + I \otimes a_1 & \mathbf{x} \\ \mathbf{z} & -I \otimes a_1^\dagger \end{pmatrix} \quad (7.1)$$

where $a \in \mathfrak{a}_2^f$, $a_1 \in \mathfrak{a}_5$, a_1^\dagger is the Hermitian conjugate of a_1 (with respect to the quaternion units), I is the 3×3 identity matrix, $\mathbf{x} \in \mathbf{C}^3 \otimes \mathbf{J}_3^4$, $\mathbf{z} \in \mathbf{C}^3 \otimes \overline{\mathbf{J}}_3^4$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \mathbf{z} = (z_1, z_2, z_3) , \quad x_i \in \mathbf{J}_3^4 \quad z_i \in \overline{\mathbf{J}}_3^4 , \quad i = 1, 2, 3$$

The commutator of two such matrices is formally the same as for \mathbf{e}_6 (cfr. (6.2)):
A few remarks are in order :

1. since $\mathbf{a}_5 \simeq \mathfrak{sl}(3, \mathbf{Q})$ (cfr. e.g. [62, 19, 63]), then $a_1 \in \mathbf{a}_5$ can be written as the sum of a skew-Hermitian matrix and a traceless Hermitian matrix in \mathbf{J}_0 , with $\mathbf{J} = \mathbf{J}_3^4$; it is worth noting that $\mathfrak{sl}(3, \mathbf{Q})$ has 35 parameters, only one less than $gl(3, \mathbf{Q})$ since the trace that is taken away from $\mathfrak{gl}(3, \mathbf{Q})$ is in \mathbf{C} , not in $\mathbf{C} \otimes \mathbf{Q}$;
2. writing $\mathbf{x}^+ \diamond \mathbf{y}^- := c \otimes I + I \otimes c_1$, we have that both c and c_1 are traceless, $c \in \mathbf{a}_2$ and $c_1 \in \mathfrak{sl}(3, \mathbf{Q})$ (and indeed this latter has 35 complex parameters), and $\mathbf{y}^- \bullet \mathbf{x}^+ = I \otimes c_1^\dagger$; according to the previous point, the trace that we take away with the term $I \otimes \frac{1}{3}t(x_i^\pm, y_i^\mp)I$ in (5.4) is in \mathbf{C} and $t(x, y) \neq t(xy)$ in general, due to non-commutativity;
3. terms like $(I \otimes a_1)\mathbf{y}^+ + \mathbf{y}^+(I \otimes a_1^\dagger)$ are in $\mathbf{C}^3 \otimes \mathbf{J}_3^4$, namely they are matrix-valued vectors with Hermitian matrix elements;
4. the correspondence between matrix elements in (7.1) and Tits' construction is similar to the one shown in (5.7) (and commented in Sec. 6), and it is omitted here;
5. the Jacobi identity is demonstrated in section 8;
6. the adjoint action in \mathbf{e}_7 implicitly provides us with the action of \mathbf{e}_6 on the fundamental representations $\mathbf{27}$ and $\overline{\mathbf{27}}$, since $\mathbf{e}_7 \simeq \mathbf{e}_6 \oplus \mathbf{C} \oplus (\mathbf{J}_3^8, \overline{\mathbf{J}}_3^8)$.

This last point deserves to be commented a little further, since it allows us to write the action of \mathbf{e}_7 by means of matrices that associate with respect to the standard matrix product instead of non-associative matrices of \mathbf{J}_3^8 . In a way, we are nothing but doubling the procedure already implemented for \mathbf{g}_2 in Sec. 4, where we have realized the octonions within a Zorn-type matrix, which was the basic structure for building up our representations. Here, we have to branch \mathbf{J}_3^8 into associative matrices, and still recover non-associativity through a non-standard matrix product.

As a first step, we consider the \mathbf{e}_6 subalgebra. We select an imaginary unit in \mathbf{Q} , say \mathbf{u}_1 , and restrict \mathbf{J}_3^4 to \mathbf{J}_3^2 accordingly. Then, we pick two \mathbf{a}_2 's inside \mathbf{a}_5 by setting $\mathbf{a}_2^\pm = \rho^\pm \mathfrak{sl}(3, \mathbf{C}) \subset \mathfrak{sl}(3, \mathbf{Q})$, and $\rho^\pm = \frac{1}{2}(1 \pm i\mathbf{u}_1)$. We thus get the following \mathbf{e}_6 subalgebra of matrices:

$$\left(\begin{array}{cc} a \otimes I + I \otimes (\rho^+ a_1^+ \oplus \rho^- a_1^-) & \mathbf{x} \\ \mathbf{z} & -I \otimes (\rho^- a_1^{+T} \oplus \rho^+ a_1^{-T}) \end{array} \right) \quad (7.2)$$

where $a_1^\pm \in \mathbf{a}_2$ and the vectors \mathbf{x}, \mathbf{z} have components $x_i \in \mathbf{J}_3^2$, $z_i \in \overline{\mathbf{J}}_3^2 (i = 1, 2, 3)$.

We now introduce the nilpotent elements $\varepsilon^\pm := \rho^\pm \mathbf{u}_2$, so that a generic quaternion can be written as $\mathbf{Q} \ni q = q_0^\pm \rho^\pm + q^\pm \varepsilon^\pm$. The Jordan pair $(\mathbf{27}, \overline{\mathbf{27}})$ reads then:

$$\left(\begin{array}{cc} I \otimes (\varepsilon^+ \eta^+ + \varepsilon^- \eta^-) & \varepsilon^+ \zeta^+ + \varepsilon^- \zeta^- \\ \varepsilon^+ \xi^+ + \varepsilon^- \xi^- & I \otimes (\varepsilon^+ \eta^{+T} + \varepsilon^- \eta^{-T}) \end{array} \right) \quad (7.3)$$

where $\eta^\pm \in \mathfrak{gl}(3)$ are complex 3×3 matrices, and $\zeta^+, \zeta^-, \xi^+, \xi^- \in \mathbf{b}_1$ are skew symmetric complex matrix-valued vectors.

As a convention, we associate the $\mathbf{27}$ with all the '+' signs in (7.3), and thus the $\overline{\mathbf{27}}$ with the '-' signs.

The only parameter left with respect to an element of \mathbf{e}_7 is the sum of the diagonal elements of type $\lambda \mathbf{u}_1 = \lambda(\rho^+ - \rho^-)$, ($\lambda \in \mathbf{C}$), which is associated to the generator \mathbf{C} in the decomposition of \mathbf{e}_7 (see point 6 above).

The action of \mathbf{e}_6 on its $\mathbf{27}$ is:

$$\left[\begin{pmatrix} a \otimes I + I \otimes \rho^\pm a_1^\pm & \mathbf{x} \\ \mathbf{z} & -I \otimes \rho^\mp a_1^\pm{}^T \end{pmatrix}, \begin{pmatrix} I \otimes \varepsilon^+ \eta & \varepsilon^+ \zeta \\ \varepsilon^+ \xi & I \otimes \varepsilon^+ \eta^T \end{pmatrix} \right] \quad (7.4)$$

$$:= \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where, for $x_i, z_i \in \mathbf{J}_3^2$, $x_i = x_{i+}\rho^+ + x_{i-}\rho^-$, $z_i = z_{i+}\rho^+ + z_{i-}\rho^-$:

$$\begin{aligned} C_{11} &= \varepsilon^+ (I \otimes (a_1^+ \eta - \eta a_1^-) - (x_{i+} \xi_i - \zeta_i z_{i-})) \\ C_{12} &= \varepsilon^+ \left((a \otimes I) \zeta + (I \otimes a_1^+) \zeta + \zeta (I \otimes a_1^+{}^T) \right. \\ &\quad \left. + \mathbf{x}_+ (I \otimes \eta^T) - (I \otimes \eta) \mathbf{x}_- + \mathbf{z} \times \xi \right) \\ C_{21} &= \varepsilon^+ \left(-\xi (a \otimes I) - (I \otimes a_1^+{}^T) \xi - \xi (I \otimes a_1^-) \right. \\ &\quad \left. + (I \otimes \eta^T) \mathbf{z} + \mathbf{z} (I \otimes \eta) + \mathbf{x} \times \zeta \right) \\ C_{22} &= \varepsilon^+ \left(I \otimes (-a_1^-{}^T \eta^T + \eta^T a_1^+{}^T) - (z_{i+} \zeta_i - \xi_i z_{i-}) \right). \end{aligned} \quad (7.5)$$

Notice that if $x \in \mathbf{J}_3^2$, $x = x_+\rho^+ + x_-\rho^-$, then $x = x^\dagger = x_+^T \rho^- + x_-^T \rho^+$ shows that $x_+^T = x_-$. Therefore $x \cdot (\varepsilon^+ \zeta) = (x_+ \zeta + \zeta x_-) \varepsilon^+$ where $(x_+ \zeta + \zeta x_-)$ is skew-symmetric. In particular, $t(x, \zeta) = 0$. It also holds that $(x_{i+} \xi_i - \zeta_i z_{i-})^T = (z_{i+} \zeta_i - \xi_i z_{i-})$, thus showing that $C_{22} = -C_{11}^\dagger$. It can also be shown that C_{12} and C_{21} are the product of ε^+ with a skew-symmetric complex matrix.

Analogous calculation can be performed for the $\overline{\mathbf{27}}$.

The action of the \mathbf{C} generator $\lambda(\rho^+ - \rho^-)$ on the $\mathbf{27}$ and on the $\overline{\mathbf{27}}$ is just a multiplication by 2λ on the $\mathbf{27}$ and by -2λ on the $\overline{\mathbf{27}}$.

We thus reproduce the well known branching rule of the adjoint of \mathbf{e}_7 with respect to its maximal and non-symmetric subalgebra $\mathbf{a}_2 \oplus \mathbf{a}_5$:

$$\mathbf{133} = (\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{35}) + (\mathbf{3}, \overline{\mathbf{15}}) + (\overline{\mathbf{3}}, \mathbf{15}). \quad (7.6)$$

8 Jacobi identity for $\mathbf{f}_4, \mathbf{e}_6, \mathbf{e}_7$

An equivalent way of proving that ϱ (given by (5.1), (6.1), (7.1)) is a representation of $\mathbf{f}_4, \mathbf{e}_6, \mathbf{e}_7$ respectively, is to directly prove the Jacobi identity for ϱ , and check that one gets the root diagram of the corresponding Lie algebra.

We consider the most general setting of \mathbf{e}_7 , which involves the Jordan algebra \mathbf{J}_3^4 , with non-commutative, but associative matrix elements. The $\varrho(\mathbf{f}_4)$ and $\varrho(\mathbf{e}_6)$ cases are obviously included as particular instances.

Recalling (7.1), we thus write:

$$\varrho(\mathbf{f}_1) = \begin{pmatrix} a \otimes I + I \otimes a_1 & A^+ \\ A^- & -I \otimes a_1^\dagger \end{pmatrix} \quad (8.1)$$

where $a \in \mathbf{a}_2^f, a_1 \in \mathbf{a}_5 \simeq \mathbf{sl}(3, \mathbf{Q})$ and A^+, A^- are three-vectors with elements in $\mathbf{J}_3^4, \overline{\mathbf{J}}_3^4$. Similarly, one can define $\varrho(\mathbf{f}_2)$ and $\varrho(\mathbf{f}_3)$, by respectively replacing $a \rightarrow b$ and $a \rightarrow c$ in (8.1), and:

$$[[\varrho(\mathbf{f}_1), \varrho(\mathbf{f}_2)], \varrho(\mathbf{f}_3)] + \text{cyclic permutations} := \begin{pmatrix} \mathfrak{J}_{11} & \mathfrak{J}_{12} \\ \mathfrak{J}_{21} & \mathfrak{J}_{22} \end{pmatrix} \quad (8.2)$$

In order for the Jacobi identity to hold for the matrix realization (8.1) of the adjoint of \mathbf{e}_7 , we have to prove that $\mathfrak{J}_{11} = \mathfrak{J}_{12} = \mathfrak{J}_{21} = \mathfrak{J}_{22} = 0$.

After some algebra, one computes :

$$\begin{aligned} \mathfrak{J}_{11} = & [[a, b], c] \otimes I + I \otimes [[a_1, b_1], c_1] + (A^+ \diamond B^- - B^+ \diamond A^-)(c \otimes I + I \otimes c_1) \\ & - (c \otimes I + I \otimes c_1)(A^+ \diamond B^- - B^+ \diamond A^-) \\ & + \left((a \otimes I)B^+ - (b \otimes I)A^+ + (I \otimes a_1)B^+ + B^+(I \otimes a_1^\dagger) \right. \\ & \left. - (I \otimes b_1)A^+ - A^+(I \otimes b_1^\dagger) + A^- \times B^- \right) \diamond C^- \\ & - C^+ \diamond \left(-B^-(a \otimes I) + A^-(b \otimes I) - (I \otimes a_1^\dagger)B^- - B^-(I \otimes a_1) \right. \\ & \left. + (I \otimes b_1^\dagger)A^- + A^-(I \otimes b_1) + A^+ \times B^+ \right) + \text{cyclic permutations} \end{aligned}$$

The first two terms of (8) vanish upon cyclic permutations because of the Jacobi identity in \mathbf{a}_2 and \mathbf{a}_5 . Let us consider then the terms in the r.h.s. of (8) containing A^+, B^-, c ; by denoting by $a_k, b_k \in \mathbf{J}_3^4$, ($k = 1, 2, 3$) the components of A^+ and B^- , respectively, one computes that:

$$\begin{aligned} & [A^+ \diamond B^-, c \otimes I] + ((c \otimes I)A^+) \diamond B^- - A^+ \diamond (B^-(c \otimes I)) \\ & = \left[\left(\frac{1}{3}t(a_i, b_i)I - t(a_i, b_j)E_{ij} \right) \otimes I, c \otimes I \right] + \left[I \otimes \left(\frac{1}{3}t(a_i, b_i)I - a_i b_i \right), c \otimes I \right] \\ & \quad + \left(\frac{1}{3}t(c_{ik}a_k, b_i)I - t(c_{ik}a_k, b_j)E_{ij} \right) \otimes I + I \otimes \left(\frac{1}{3}t(c_{ik}a_k, b_i)I - c_{ik}a_k b_i \right) \\ & \quad - \left(\frac{1}{3}t(a_i, b_k c_{ki})I - t(a_i, b_k c_{kj})E_{ij} \right) \otimes I + I \otimes \left(\frac{1}{3}t(a_i, b_k c_{ki})I - a_i b_k c_{ki} \right) \\ & = (-t(a_i, b_j)E_{ij}c + t(a_i, b_j)cE_{ij} - t(c_{ik}a_k, b_j)E_{ij} + t(a_i, b_k c_{kj})E_{ij}) \otimes I \\ & = (-t(a_i, b_k)c_{kj} + t(a_k, b_j)c_{ik} - t(a_k, b_j)c_{ik} + t(a_i, b_k)c_{kj})E_{ij} \otimes I \\ & = 0. \end{aligned}$$

Next, we consider the terms in the r.h.s. of (8) containing A^+, B^-, c_1 . They read:

$$\begin{aligned} & [A^+ \diamond B^-, I \otimes c_1] + ((I \otimes c_1)A^+ + A^+(I \otimes c_1^\dagger)) \diamond B^- \\ & - A^+ \diamond ((I \otimes c_1^\dagger)B^- + B^-(I \otimes c_1)) \\ & = I \otimes (c_1 a_i b_i - a_i b_i c_1) + \left(\frac{1}{3}t(c_1 a_i + a_i c_1^\dagger, b_i)I - t(c_1 a_i + a_i c_1^\dagger, b_j)E_{ij} \right) \otimes I \\ & \quad + I \otimes \left(\frac{1}{3}t(c_1 a_i + a_i c_1^\dagger, b_i)I - (c_1 a_i + a_i c_1^\dagger) b_i \right) \\ & \quad - \left(\frac{1}{3}t(a_i, c_1^\dagger b_i + b_i c_1)I - t(a_i, c_1^\dagger b_j + b_j c_1)E_{ij} \right) \otimes I \\ & \quad - I \otimes \left(\frac{1}{3}t(a_i, c_1^\dagger b_i + b_i c_1)I - a_i (c_1^\dagger b_i + b_i c_1) \right) \\ & = I \otimes \left[\frac{2}{3} \left(t(c_1 a_i + a_i c_1^\dagger, b_i) - t(a_i, c_1^\dagger b_i + b_i c_1) \right) I \right. \\ & \quad \left. - \left(t(c_1 a_i + a_i c_1^\dagger, b_j) - t(a_i, c_1^\dagger b_j + b_j c_1) \right) E_{ij} \right] \end{aligned}$$

In order to prove that the r.h.s. of (8) is zero, we write $\mathbf{sl}(\mathbf{3}, \mathbf{Q}) \ni c_1 = h + s$, where $h \in \mathbf{J}_3^4$ is Hermitian, and s skew-Hermitian (with respect to quaternion conjugation). Note that the action $x \rightarrow sx + xs^\dagger = sx - xs$ is a derivation in \mathbf{J}_3^4 . Therefore, by exploiting the identities [37, 59]:

$$\begin{aligned} t(x, y \cdot z) &= t(z, x \cdot y) \\ t(Dx, y) + t(x, Dy) &= 0 \quad \text{where } D \text{ is a derivation in } \mathbf{J}_3^4 \end{aligned} \tag{8.3}$$

one proves that the terms under consideration in the r.h.s. of (8) sum up to zero.

Finally, we consider terms in the r.h.s. of (8) which contain structures like $(A^- \times B^-) \diamond C^-$; they read:

$$\begin{aligned} & (A^- \times B^-) \diamond C^- + (B^- \times C^-) \diamond A^- + (C^- \times A^-) \diamond B^- \\ & = \epsilon_{i\ell k} \left(\frac{1}{3}t(a_\ell \# b_k, c_i)I - t(a_\ell \# b_k, c_j)E_{ij} \right) \otimes I \\ & \quad + I \otimes \epsilon_{i\ell k} \left(\frac{1}{3}t(a_\ell \# b_k, c_i)I - (a_\ell \# b_k) c_i \right) + \text{cyclic permutations} \\ & := M^{(1)} \otimes I + I \otimes M^{(2)}. \end{aligned} \tag{8.4}$$

In order to show that $M^{(1)} = M^{(2)} = 0$, we observe, after [37], that $t(a\#b, c)$ is symmetric in a, b, c . Let us consider $M^{(1)}$ first. For $i \neq j$, then either $j = \ell$ or $j = k$. The coefficient of E_{ij} is therefore :

$$\begin{aligned} & \epsilon_{ijk} (t(a_j\#b_k, c_j) - t(a_k\#b_j, c_j) + t(b_j\#c_k, a_j) - t(b_k\#c_j, a_j) \\ & + t(c_j\#a_k, b_j) - t(c_k\#a_j, b_j)) = 0 \end{aligned}$$

For $i = j$, by summing over i, ℓ, k and using the notation $\tau_{\ell ki} := t(a_\ell\#b_k, c_i) + t(b_\ell\#c_k, a_i) + t(c_\ell\#a_k, b_i)$, one can easily check that:

$$\epsilon_{1\ell k}\tau_{\ell k1} = \epsilon_{2\ell k}\tau_{\ell k2} = \epsilon_{3\ell k}\tau_{\ell k3} := \omega$$

. Thus :

$$\epsilon_{i\ell k}\tau_{\ell ki}(\frac{1}{3}I - E_{ii}) = \omega(I - E_{11} - E_{22} - E_{33}) = 0$$

This proves that $M^{(1)} = 0$. For what concerns $M^{(2)}$, we observe that $\frac{1}{3}t(x\#y, z)I - (x\#y)z + \{\text{cyclic permutations}\}$ is linear and symmetric in x, y, z . It is indeed the polarization of (2.6), hence it is zero, implying that $M^{(2)} = 0$. We stress that it is crucial to have associativity with respect to the standard matrix product of elements in \mathbf{J}_3^4 , in order to apply the polarization statement; we do need in particular $x^2x = xx^2 = x^2 \cdot x$, which does indeed hold in the associative case.

Analogous calculations for the other terms in the r.h.s. of (8) involving $\{B^+, A^-, c\}$, $\{B^+, A^-, c_1\}$, $\{B^+, A^+, C^+\}$ plus their cyclic permutations prove that $\mathfrak{J}_{11} = 0$.

Next, we proceed to consider \mathfrak{J}_{12} which, after some algebra, can be computed to read :

$$\begin{aligned} \mathfrak{J}_{12} = & ([a, b] \otimes I + I \otimes [a_1, b_1] + A^+ \diamond B^- - B^+ \diamond A^-) C^+ \\ & - \left((a \otimes I)B^+ - (b \otimes I)A^+ + (I \otimes a_1)B^+ + B^+(I \otimes a_1^\dagger) \right. \\ & \left. - (I \otimes b_1)A^+ - A^+(I \otimes b_1^\dagger) + A^- \times B^- \right) (I \otimes c_1^\dagger) \\ & - (c \otimes I + I \otimes c_1) \left((a \otimes I)B^+ - (b \otimes I)A^+ + (I \otimes a_1)B^+ + B^+(I \otimes a_1^\dagger) \right. \\ & \left. - (I \otimes b_1)A^+ - A^+(I \otimes b_1^\dagger) + A^- \times B^- \right) \\ & - C^+ \left(I \otimes [a_1^\dagger, b_1^\dagger] + A^- \bullet B^+ - B^- \bullet A^+ \right) \\ & + \left(-B^-(a \otimes I) + A^-(b \otimes I) - (I \otimes a_1^\dagger)B^- - B^-(I \otimes a_1) \right. \\ & \left. + (I \otimes b_1^\dagger)A^- + A^-(I \otimes b_1) + A^+ \times B^+ \right) \times C^- + \text{cyclic permutations.} \end{aligned}$$

Many terms cancel out trivially, and one remains with terms of the following three types:

$$\begin{aligned} 1) & (I \otimes c_1)(A^- \times B^-) + (A^- \times B^-)(I \otimes c_1^\dagger) \\ & + ((I \otimes c_1^\dagger)A^-) + (A^-(I \otimes c_1)) \times B^- - ((I \otimes c_1^\dagger)B^-) + (B^-(I \otimes c_1)) \times A^- \\ 2) & - (c \otimes I)(A^- \times B^-) - (A^-(c \otimes I)) \times B^- + (B^-(c \otimes I)) \times A^- \\ 3) & (A^+ \times B^+) \times C^- + (B^+ \diamond C^-)A^+ - (A^+ \diamond C^-)B^+ \\ & + A^+(C^- \bullet B^+) - B^+(C^- \bullet A^+), \end{aligned}$$

where we remark that the first two terms show the action of the \mathbf{a}_5 and \mathbf{a}_2 subalgebras as derivations.

Let us analyze each of the terms 1) - 3) separately.

1) Writing this term explicitly, one obtains:

$$\begin{aligned} & \epsilon_{ijk}(c_1(a_j \# b_k) + (a_j \# b_k)c_1^\dagger) + \epsilon_{ijk}(c_1^\dagger a_j + a_j c_1) \# b_k - \epsilon_{ikj}(c_1^\dagger b_k + b_k c_1) \# a_j \\ &= \epsilon_{ijk} \left[c_1(a_j \# b_k) + (a_j \# b_k)c_1^\dagger + (c_1^\dagger a_j + a_j c_1) \# b_k + (c_1^\dagger b_k + b_k c_1) \# a_j \right] \end{aligned}$$

In order to show that the expression in brackets is identically zero, we write $\mathfrak{sl}(\mathbf{3}, \mathbf{Q}) \ni c_1 = h + s$, namely as the sum of a traceless Hermitian matrix h and of a skew-Hermitian matrix s . Since the expression under consideration is linear in c_1 , we can consider the two contributions of h and s separately. The contribution of h reads:

$$\begin{aligned} & 4((a \cdot b) \cdot h + (a \cdot h) \cdot b + (b \cdot h) \cdot a) - 4(t(a)b \cdot h + t(b)h \cdot a + t(h)a \cdot b) \\ & - 2[(t(b, h) - t(b)t(h))a + (t(h, a) - t(h)t(a))b + (t(a, b) - t(a)t(b))h] \\ & - 2(t(a, b \cdot h) + t(a \cdot h, b) - t(a, h)t(b) - t(b, h)t(a) - t(a, b)t(h) + t(a)t(b)t(h))I, \end{aligned}$$

where we have added all terms in $t(h)$, since h is traceless. By adopting the first identity in (8.3) we see that we have obtained a symmetric multilinear form that is (12 times) the polarization of (2.6), hence it is zero.

On the other hand, the contribution of s can be easily shown to be zero, because $sx - xs$ is a derivation for $x \in \mathbf{J}_3^4$; we have indeed that if D is a derivation $t(Dx) = 0$ and $t(D(a), b) + t(a, D(b)) = 0$. Hence:

$$D(a \# b) = 2D(a) \cdot b + 2a \cdot D(b) - D(a)t(b) - D(b)t(a) = D(a) \# b + a \# D(b), \quad (8.5)$$

and this implies the vanishing of the contribution of s to term reported at point 1.

2) We can write this expression as:

$$-(\epsilon_{ijk}c_{\ell i} + \epsilon_{lik}c_{ji} - \epsilon_{lij}c_{ki})(a_j \# b_k)$$

For $\ell = k$, or $\ell = j$, or $k = j$, the first round bracket trivially vanishes. For $\ell \neq j \neq k \neq \ell$, it can be written as $\epsilon_{\ell jk}c_{\ell \ell} + \epsilon_{\ell jk}c_{jj} + \epsilon_{\ell jk}c_{kk} = 0$, since $t(c) = 0$.

3) Explicit calculation shows that the i -th component of this term reads:

$$\begin{aligned} & (a_\ell \# b_i) \# c_\ell - (a_i \# b_\ell) \# c_\ell + t(b_\ell, c_\ell)a_i + t(a_i, c_\ell)b_\ell - t(a_\ell, c_\ell)b_i - t(b_i, c_\ell)a_\ell \\ & - a_i c_\ell b_\ell - b_\ell c_\ell a_i + a_\ell c_\ell b_i + b_i c_\ell a_\ell \\ & = -\{a_\ell, c_\ell, b_i\} + \{a_i, c_\ell, b_\ell\} - a_i c_\ell b_\ell - b_\ell c_\ell a_i + a_\ell c_\ell b_i + b_i c_\ell a_\ell = 0, \end{aligned}$$

where the triple product $\{x, y, z\} := V_{x,y}z$ has been introduced in section 2 and, in the associative case we are considering here: $\{x, y, z\} = xyz + zy x$, thus implying that also the term 3) vanishes.

This ends the proof of the fact that $\mathfrak{J}_{12} = 0$.

Analogous calculations show that also $\mathfrak{J}_{21} = \mathfrak{J}_{22} = 0$, thus proving the Jacobi identity for the matrix realization ϱ (7.1) of the adjoint of \mathbf{e}_7 , implying the Jacobi identity for the matrix realizations ϱ (5.1) and (6.1) of the adjoint of \mathbf{f}_4 and \mathbf{e}_6 , respectively. ■

9 $\mathfrak{n} = 8$: Matrix representation of \mathbf{e}_8

Finally, we consider the case of \mathbf{e}_8 , the largest finite-dimensional exceptional Lie algebra.

We use the notation $L_x z := x \cdot z$ and, for $\mathbf{x} \in \mathbf{C}^3 \otimes \mathbf{J}_3^8$ with components (x_1, x_2, x_3) , $L_{\mathbf{x}} \in \mathbf{C}^3 \otimes L_{\mathbf{J}_3^8}$ denotes the corresponding operator-valued vector with components $(L_{x_1}, L_{x_2}, L_{x_3})$.

We can write an element a_1 of \mathbf{e}_6 as $a_1 = L_x + \sum [L_{x_i}, L_{y_i}]$ where $x, x_i, y_i \in \mathbf{J}_3^8$ ($i = 1, 2, 3$) and $t(x) = 0$, [54] [59]. The adjoint is defined by $a_1^\dagger := L_x - [L_{x_1}, L_{x_2}]$. Notice that the operators

$F := [L_{x_i}, L_{y_i}]$ span the \mathfrak{f}_4 subalgebra of \mathfrak{e}_6 , namely the derivation algebra of \mathbf{J}_3^8 (recall that the Lie algebra of the structure group of \mathbf{J}_3^8 is $\mathfrak{e}_6 \oplus \mathbf{C}$).

We should remark that $(a_1, -a_1^\dagger)$ is a derivation in the Jordan Pair $(\mathbf{J}_3^8, \overline{\mathbf{J}}_3^8)$, and it is here useful to recall the relationship between the structure group of a Jordan algebra J and the automorphism group of a Jordan Pair $V = (J, J)$ goes as follows [45]: if $g \in \text{Str}(J)$ then $(g, U_{g(I)}^{-1}g) \in \text{Aut}(V)$. In our case, for $g = 1 + \epsilon(L_x + F)$, at first order in ϵ we get (namely, in the tangent space of the corresponding group manifold) $U_{g(I)}^{-1}g = 1 + \epsilon(-L_x + F) + O(\epsilon^2)$.

Next, we introduce a product $\star, [6]$, such that $L_x \star L_y := L_{x \cdot y} + [L_x, L_y]$, $F \star L_x := 2FL_x$ and $L_x \star F := 2L_x F$ for $x, y \in \mathbf{J}_3^8$, including each component x of $\mathbf{x} \in \mathbf{C}^3 \otimes \mathbf{J}_3^8$ and y of $\mathbf{y} \in \mathbf{C}^3 \otimes \mathbf{J}_3^8$. By denoting with $[\ ; \]$ the commutator with respect to the \star product, we also require that $[F_1; F_2] := 2[F_1, F_2]$. One thus obtains that $L_x \star L_y + L_y \star L_x = 2L_{x \cdot y}$ and $[F; L_x] := F \star L_x - L_x \star F = 2[F, L_x] = 2L_{F(x)}$, where the last equality holds because F is a derivation in \mathbf{J}_3^8 .

Therefore, for $\mathfrak{f} \in \mathfrak{e}_8$, we write:

$$\varrho(\mathfrak{f}) = \begin{pmatrix} a \otimes Id + I \otimes a_1 & L_{\mathbf{x}^+} \\ L_{\mathbf{x}^-} & -I \otimes a_1^\dagger \end{pmatrix} \quad (9.1)$$

where $a \in \mathfrak{a}_2^{\mathfrak{C}}$, $a_1 \in \mathfrak{e}_6$, and we recall that I is the 3×3 identity matrix, as above; furthermore, $Id := L_I$ is the identity operator in $L_{\mathbf{J}_3^8}$ (namely, $L_I L_x = L_x$). Notice that Id is the identity also with respect to the \star product.

By extending the \star product in an obvious way to the matrix elements (9.1), one achieves that $(I \otimes a_1) \star L_{\mathbf{y}^+} + L_{\mathbf{y}^+} \star (I \otimes a_1^\dagger) = 2L_{(I \otimes a_1)\mathbf{y}^+}$ and $(I \otimes a_1^\dagger) \star L_{\mathbf{y}^-} + L_{\mathbf{y}^-} \star (I \otimes a_1) = 2L_{(I \otimes a_1^\dagger)\mathbf{y}^-}$.

After some algebra, the commutator of two matrices like (9.1) can be computed to read :

$$\begin{aligned} & \left[\begin{pmatrix} a \otimes Id + I \otimes a_1 & L_{\mathbf{x}^+} \\ L_{\mathbf{x}^-} & -I \otimes a_1^\dagger \end{pmatrix}, \begin{pmatrix} b \otimes Id + I \otimes b_1 & L_{\mathbf{y}^+} \\ L_{\mathbf{y}^-} & -I \otimes b_1^\dagger \end{pmatrix} \right] \\ & := \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \end{aligned} \quad (9.2)$$

where:

$$\begin{aligned} C_{11} &= [a, b] \otimes Id + 2I \otimes [a_1, b_1] + L_{\mathbf{x}^+} \diamond L_{\mathbf{y}^-} - L_{\mathbf{y}^+} \diamond L_{\mathbf{x}^-} \\ C_{12} &= (a \otimes Id)L_{\mathbf{y}^+} - (b \otimes Id)L_{\mathbf{x}^+} + 2L_{(I \otimes a_1)\mathbf{y}^+} \\ & \quad - 2L_{(I \otimes b_1)\mathbf{x}^+} + L_{\mathbf{x}^-} \times L_{\mathbf{y}^-} \\ C_{21} &= -L_{\mathbf{y}^-}(a \otimes Id) + L_{\mathbf{x}^-}(b \otimes Id) - 2L_{(I \otimes a_1^\dagger)\mathbf{y}^-} \\ & \quad + 2L_{(I \otimes b_1^\dagger)\mathbf{x}^-} + L_{\mathbf{x}^+} \times L_{\mathbf{y}^+} \\ C_{22} &= 2I \otimes [a_1^\dagger, b_1^\dagger] + L_{\mathbf{x}^-} \bullet L_{\mathbf{y}^+} - L_{\mathbf{y}^-} \bullet L_{\mathbf{x}^+}. \end{aligned} \quad (9.3)$$

It should be stressed that the products occurring in (9.3) do differ from those of (5.4); namely,

they are defined as follows⁷ :

$$\begin{aligned}
L_{\mathbf{x}^+} \diamond L_{\mathbf{y}^-} &:= \left(\frac{1}{3}t(x_i^+, y_i^-)I - t(x_i^+, y_j^-)E_{ij} \right) \otimes Id + \\
&\quad I \otimes \left(\frac{1}{3}t(x_i^+, y_i^-)Id - L_{x_i^+ \cdot y_i^-} - [L_{x_i^+}, L_{y_i^-}] \right) \\
L_{\mathbf{x}^-} \bullet L_{\mathbf{y}^+} &:= I \otimes \left(\frac{1}{3}t(x_i^-, y_i^+)Id - L_{x_i^- \cdot y_i^+} - [L_{x_i^-}, L_{y_i^+}] \right) \\
L_{\mathbf{x}^\pm} \times L_{\mathbf{y}^\pm} &:= L_{\mathbf{x}^\pm \times \mathbf{y}^\pm} = L_{\epsilon_{ijk}(x_j^\pm \# y_k^\pm)}.
\end{aligned} \tag{9.4}$$

From the properties of the triple product of Jordan algebras (discussed in Sec. 2), it holds that $L_{x_i^+ \cdot y_i^-} + [L_{x_i^+}, L_{y_i^-}] = \frac{1}{2}V_{x_i^+, y_i^-} \in \mathbf{e}_6 \oplus \mathbf{C}$, see (2.7). Moreover, one can readily check that $[a_1^\dagger, b_1^\dagger] = -[a_1, b_1]^\dagger$, $(a \otimes Id)L_b = L_{(a \otimes Id)b}$ and $L_{\mathbf{y}^-} \bullet L_{\mathbf{x}^+} = I \otimes \left(\frac{1}{3}t(x_i^+, y_i^-)Id - L_{x_i^+ \cdot y_i^-} - [L_{x_i^+}, L_{y_i^-}] \right)^\dagger$; this result implies that we are actually considering an algebra.

In the next section we are going to prove that Jacobi's identity holds for the algebra of Zorn-type matrices (9.1), with Lie product given by (9.2) - (9.4). On the other hand, once Jacobi's identity is proven, the fact that the Lie algebra so represented is \mathbf{e}_8 is made obvious by a comparison with the root diagram in figure 1, for $n = 8$; in this case, we have:

- 1) an $g_0^8 = \mathbf{e}_6$, commuting with \mathbf{a}_2^8 ;
- 2) As in general, the three Jordan Pairs which globally transform as a $(\mathbf{3}, \overline{\mathbf{3}})$ of \mathbf{a}_2^8 ; in this case, each of them transforms as a $(\mathbf{27}, \overline{\mathbf{27}})$ of \mathbf{e}_6 .

As a consequence, we reproduce the well known branching rule of the adjoint of \mathbf{e}_8 with respect to its maximal and non-symmetric subalgebra $\mathbf{a}_2^8 \oplus \mathbf{e}_6$:

$$248 = (\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{78}) + (\mathbf{3}, \mathbf{27}) + (\overline{\mathbf{3}}, \overline{\mathbf{27}}). \tag{9.5}$$

10 Jacobi identity for \mathbf{e}_8

We use the same notation as in section 8, and write (9.1) in a slight different way, namely, for for $\mathbf{f}_1 \in \mathbf{e}_8$:

$$\varrho(\mathbf{f}_1) = \begin{pmatrix} a \otimes I + I \otimes a_1 & A^+ \\ A^- & -I \otimes a_1^\dagger \end{pmatrix}, \tag{10.1}$$

where $a \in \mathbf{a}_2^8$, $a_1 \in \mathbf{e}_6$ and A^+, A^- three vectors with elements in $\mathbf{J}_3^8, \overline{\mathbf{J}}_3^8$. Similarly, one can define $\varrho(\mathbf{f}_2)$ and $\varrho(\mathbf{f}_3)$ by respectively replacing $a \rightarrow b$ and $a \rightarrow c$ in (10.1). Let us then write:

$$[[\varrho(\mathbf{f}_1), \varrho(\mathbf{f}_2)], \varrho(\mathbf{f}_3)] + \text{cyclic permutations} := \begin{pmatrix} \mathfrak{J}_{11} & \mathfrak{J}_{12} \\ \mathfrak{J}_{21} & \mathfrak{J}_{22} \end{pmatrix} \tag{10.2}$$

In order for the Jacobi identity to hold for the matrix realization (10.1) of the adjoint of \mathbf{e}_8 , we have to prove that $\mathfrak{J}_{11} = \mathfrak{J}_{12} = \mathfrak{J}_{21} = \mathfrak{J}_{22} = 0$.

⁷It should be stressed here that the matrix products $x \diamond y$, $x \cdot y$ e $x \times y$ defined in (9.4), never appeared (to the best of our present knowledge) in the literature, and are an original result of the present investigation.

After some algebra, we compute:

$$\begin{aligned}
\mathfrak{J}_{11} = & [[a, b], c] \otimes Id + 4I \otimes [[a_1, b_1], c_1] + (A^+ \diamond B^- - B^+ \diamond A^-)(c \otimes Id + I \otimes c_1) \\
& - (c \otimes Id + I \otimes c_1)(A^+ \diamond B^- - B^+ \diamond A^-) \\
& + \left((a \otimes Id)B^+ - (b \otimes Id)A^+ + (I \otimes a_1)B^+ + B^+(I \otimes a_1^\dagger) \right. \\
& \left. - (I \otimes b_1)A^+ - A^+(I \otimes b_1^\dagger) + A^- \times B^- \right) \diamond C^- \\
& - C^+ \diamond \left(-B^-(a \otimes Id) + A^-(b \otimes Id) - (I \otimes a_1^\dagger)B^- - B^-(I \otimes a_1) \right. \\
& \left. + (I \otimes b_1^\dagger)A^- + A^-(I \otimes b_1) + A^+ \times B^+ \right) + \text{cyclic permutations}
\end{aligned}$$

The first two terms in the r.h.s. of (10) vanish upon cyclic permutations, because of the Jacobi identity in $\mathfrak{a}_2^{\mathfrak{s}}$ and \mathfrak{e}_6 . The terms containing A^+, B^-, c can be proved to vanish, by the very same arguments used in section 8.

Next, we consider the terms containing A^+, B^-, c_1 . By denoting with $a_k, b_k \in \mathbf{J}_3^{\mathfrak{s}}$ ($k = 1, 2, 3$), the components of A^+ and B^- respectively, and using the shorthand notation: $E(x, y) := L_{x \cdot y} + [L_x, L_y] = \frac{1}{2}V_{x, y}$, for $x, y \in \mathbf{J}_3^{\mathfrak{s}}$, one can compute that:

$$\begin{aligned}
& [A^+ \diamond B^-; I \otimes c_1] + ((I \otimes c_1)A^+ + A^+(I \otimes c_1^\dagger)) \diamond B^- \\
& - A^+ \diamond ((I \otimes c_1^\dagger)B^- + B^-(I \otimes c_1)) \\
& = 2I \otimes [c_1, E(a_i, b_i)] + 2 \left(\frac{1}{3}t(c_1(a_i), b_i)I - t(c_1(a_i), b_j)E_{ij} \right) \otimes Id \\
& \quad + 2I \otimes \left(\frac{1}{3}t(c_1(a_i), b_i)Id - E(c_1(a_i), b_i) \right) \\
& \quad - 2 \left(\frac{1}{3}t(a_i, c_1^\dagger(b_i))I - t(a_i, c_1^\dagger(b_j))E_{ij} \right) \otimes Id \\
& \quad - 2I \otimes \left(\frac{1}{3}t(a_i, c_1^\dagger(b_i))Id - E(a_i, c_1^\dagger(b_i)) \right) \\
& = \left(\frac{4}{3}(t(c_1(a_i), b_i) - t(a_i, c_1^\dagger(b_i)))I - 2(t(c_1(a_i), b_j) - t(a_i, c_1^\dagger(b_j)))E_{ij} \right) \otimes Id \\
& \quad + 2I \otimes \left([c_1, E(a_i, b_i)] - E(c_1(a_i), b_i) + E(a_i, c_1^\dagger(b_i)) \right).
\end{aligned}$$

In order to prove that (10) sums up to zero, we start and observe that $t(c_1(a), b) = t(a, c_1^\dagger(b))$; this is easily shown by writing $c_1 = L_x + F$ (hence $c_1^\dagger = L_x - F$) and noticing that $t(L_x a, b) = t(x \cdot a, b) = t(a, x \cdot b) = t(a, L_x(b))$ and $t(Fx, y) + t(x, Fy) = 0$, being F is a derivation in $\mathbf{J}_3^{\mathfrak{s}}$. Moreover, $[c_1, E(a, b)] = E(c_1(a), b) - E(a, c_1^\dagger(b))$, by (2.10). This indeed implies that (10) vanishes.

Finally, we consider terms in the r.h.s. of (10) which contain structures like $(A^- \times B^-) \diamond C^-$; they read:

$$\begin{aligned}
& (A^- \times B^-) \diamond C^- + (B^- \times C^-) \diamond A^- + (C^- \times A^-) \diamond B^- \\
& = \epsilon_{i\ell k} \left(\frac{1}{3}t(a_\ell \# b_k, c_i)I - t(a_\ell \# b_k, c_j)E_{ij} \right) \otimes Id \\
& \quad + I \otimes \epsilon_{i\ell k} \left(\frac{1}{3}t(a_\ell \# b_k, c_i)Id - E(a_\ell \# b_k, c_i) \right) + \text{cyclic permutations} \\
& := M^{(1)} \otimes I + I \otimes M^{(2)}.
\end{aligned} \tag{10.3}$$

$M^{(1)} = 0$, by the same argument used in section 8. Let us here show that $M^{(2)} = 0$. In order to do this, we write $(a_j, b_k, c_i) := \frac{1}{3}t(a_j \# b_k, c_i)I - (a_j \# b_k) \cdot c_i$. Thence:

$$M^{(2)} = \epsilon_{ijk} (L_{(a_j, b_k, c_i)} - [L_{a_j \# b_k}, L_{c_i}]) + \text{cyclic permutations} \tag{10.4}$$

For each fixed i, j, k , it holds that

$$\begin{aligned}
& \epsilon_{ijk}(a_j, b_k, c_i) + \epsilon_{jki}(b_k, c_i, a_j) + \epsilon_{kij}(c_i, a_j, b_k) \\
& = \epsilon_{ijk}((a_j, b_k, c_i) + (b_k, c_i, a_j) + (c_i, a_j, b_k))
\end{aligned} \tag{10.5}$$

Since (x, y, z) is symmetric in x, y and linear in x, y, z , the above expression is linear and symmetric in a_j, b_k, c_i , thus it is the polarization of $(x, x, x) = 2(\frac{1}{3}t(x^\#, x)I - x^\# \cdot x) = 0$, by (2.6). Similarly, for $[L_{a_j \# b_k}, L_{c_i}] + \text{cyclic permutations}$, we get the polarization of $[L_{x^\#}, L_x]$, which is zero by the Jordan identity (2.5), namely:

$$[L_{x^\#}, L_x]z = x^\# \cdot (x \cdot z) - x \cdot (x^\# \cdot z) = x^2 \cdot (x \cdot z) - x \cdot (x^2 \cdot z) = 0 \quad \forall z \in J \quad (10.6)$$

Analogous calculations for terms in the r.h.s. of (10) which contain structures like $\{B^+, A^-, c\}$, $\{B^+, A^-, c_1\}$, $\{B^+, A^+, C^+\}$ (plus their cyclic permutations) prove that $\mathfrak{J}_{11} = 0$.

Next, we proceed to consider \mathfrak{J}_{12} which, after some algebra, can be computed to read :

$$\begin{aligned} \mathfrak{J}_{12} = & ([a, b] \otimes I + 2I \otimes [a_1, b_1] + A^+ \diamond B^- - B^+ \diamond A^-) C^+ \\ & - \left((a \otimes I)B^+ - (b \otimes I)A^+ + (I \otimes a_1)B^+ + B^+(I \otimes a_1^\dagger) \right. \\ & \left. - (I \otimes b_1)A^+ - A^+(I \otimes b_1^\dagger) + A^- \times B^- \right) (I \otimes c_1^\dagger) \\ & - (c \otimes I + I \otimes c_1) \left((a \otimes I)B^+ - (b \otimes I)A^+ + (I \otimes a_1)B^+ + B^+(I \otimes a_1^\dagger) \right. \\ & \left. - (I \otimes b_1)A^+ - A^+(I \otimes b_1^\dagger) + A^- \times B^- \right) \\ & - C^+ \left(I \otimes [a_1^\dagger, b_1^\dagger] + A^- \bullet B^+ - B^- \bullet A^+ \right) \\ & + \left(-B^-(a \otimes I) + A^-(b \otimes I) - (I \otimes a_1^\dagger)B^- - B^-(I \otimes a_1) \right. \\ & \left. + (I \otimes b_1^\dagger)A^- + A^-(I \otimes b_1) + A^+ \times B^+ \right) \times C^- + \text{cyclic permutations} \end{aligned}$$

By noticing that $[a_1^\dagger, b_1^\dagger] = -[a_1, b_1]^\dagger$ and that, as already noticed, $(a \otimes Id)L_b = L_{(a \otimes Id)b}$, as already noticed, one finds that many terms cancel out trivially, and only terms of the following three types remain:

$$\begin{aligned} 1) \quad & -2L_{(I \otimes c_1)a^- \times b^-} - L_{(I \otimes c_1^\dagger)a^- \times b^-} + L_{(I \otimes c_1^\dagger)b^- \times a^-} \\ 2) \quad & - (c \otimes I)L_{a^- \times b^-} - L_{a^-(c \otimes I)} \times b^- + L_{b^-(c \otimes I)} \times a^- \\ 3) \quad & L_{(a^+ \times b^+) \times c^-} + t(b_i^+, c_i^-)L_{a^+} - t(b_i^+, c_j^-)E_{ij} \otimes IdL_{a^+} \\ & - t(a_i^+, c_i^-)L_{b^+} + t(a_i^+, c_j^-)E_{ij} \otimes IdL_{b^+} - L_{V_{b_i^+, c_i^+}a^+} + L_{V_{a_i^+, c_i^+}b^+} \end{aligned}$$

Terms like 1) and 2) can be shown to vanish using similar arguments to those of section 8. The i -th component of terms like 3) can be written as (omitting the $+$, $-$ superscripts):

$$\begin{aligned} & L_{(a_\ell \# b_i) \# c_\ell} - L_{(a_i \# b_\ell) \# c_\ell} + t(b_\ell, c_\ell)L_{a_i} + t(a_i, c_\ell)L_{b_\ell} \\ & - t(a_\ell, c_\ell)L_{b_i} - t(b_i, c_\ell)L_{a_\ell} - L_{V_{b_\ell, c_\ell}a_i} + L_{V_{a_\ell, c_\ell}b_i}, \end{aligned} \quad (10.7)$$

which vanishes because of (2.7).

This ends the proof of the fact that $\mathfrak{J}_{12} = 0$.

Analogous calculations show that also $\mathfrak{J}_{21} = \mathfrak{J}_{22} = 0$, thus proving the Jacobi identity for the matrix realization (10.1) (or, equivalently (9.1)) of the adjoint of \mathbf{e}_8 . ■

11 Future developments

There are several topics that we are planning to develop in the future.

One is the extension of the Zorn-type representations to the Lie algebra of the semi-direct product group $E_{7\frac{1}{2}}$, through a representation of the *sextions* [64, 65] and of the algebra of their derivations.

A second interesting venue of developments is the characterization of all real forms of these representations of the exceptional Lie algebras, as well as the treatment of *split* forms of Hurwitz's algebras \mathbf{C} , \mathbf{Q} , \mathbf{C} , with a particular attention to the coset spaces related to the scalar manifolds in supergravity. This would yield a *Zorn-like* realization of (some of) the maximal non-symmetric embeddings considered in [39], and proved in a broader framework in [43].

Moreover, it would be interesting to consider Jordan pairs for *semi-simple* Jordan algebras of rank 3 of relevance for supergravity theories, along the lines of the treatment given in [39].

We plan then to proceed to the study of the representations of quantum exceptional groups - in particular *quantum* \mathbf{e}_8 - and of integrable models built on them. We aim at a new perspective of elementary particle physics at the early stages of the Universe based on the idea that interactions, defined in a purely algebraic way, are the fundamental objects of the theory, whereas space-time, hence gravity, are derived structures.

Acknowledgments

The work of AM is supported in part by the FWO - Vlaanderen, Project No. G.0651.11, and in part by the Interuniversity Attraction Poles Programme initiated by the Belgian Science Policy (P7/37).

The work of PT is supported in part by the *Istituto Nazionale di Fisica Nucleare* grant In. Spec. GE 41.

References

- [1] P. Ramond, *Exceptional Groups and Physics*, Plenary Talk delivered at the Conference Groupe 24, Paris, July 2002, [arXiv:hep-th/0301050v1](#).
- [2] E. Cremmer and B. Julia, *The $\mathcal{N}=8$ Supergravity Theory. 1. The Lagrangian*, Phys. Lett. **B80**, 48 (1978). E. Cremmer and B. Julia, *The $SO(8)$ Supergravity*, Nucl. Phys. **B159**, 141 (1979).
- [3] C. Hull and P. K. Townsend, *Unity of Superstring Dualities*, Nucl. Phys. **B438**, 109 (1995), [hep-th/9410167](#).
- [4] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, 1978, Academic Press.
- [5] P. Truini, G. Olivieri, L.C. Biedenharn, *The Jordan Pair Content Of The Magic Square And The Geometry Of The Scalars In $\mathcal{N}=2$ Supergravity*, Lett. Math. Phys. **9**, 255 (1985).
- [6] P. Truini, G. Olivieri, L.C. Biedenharn, *Three Graded Exceptional Algebras And Symmetric Spaces*, Z. Phys. **C33**, 47 (1986).
- [7] S. Ferrara and A. Marrani, *Symmetric Spaces in Supergravity*, in : “Symmetry in Mathematics and Physics” (D. Babbitt, V. Vyjayanthi and R. Fioresi Eds.), Contemporary Mathematics **490**, American Mathematical Society (Providence RI, 2009), [arXiv:0808.3567 \[hep-th\]](#).
- [8] S. Ferrara, R. Kallosh and A. Strominger, *$\mathcal{N}=2$ extremal black holes*, Phys. Rev. **D52** (1995) 5412, [hep-th/9508072](#). A. Strominger, *Macroscopic entropy of $\mathcal{N}=2$ extremal black holes*, Phys. Lett. **B383**, 39 (1996), [hep-th/9602111](#). S. Ferrara and R. Kallosh, *Supersymmetry and attractors*, Phys. Rev. **D54**, 1514 (1996), [hep-th/9602136](#). S. Ferrara

- and R. Kallosh, *Universality of supersymmetric attractors*, Phys. Rev. **D54**, 1525 (1996), [hep-th/9603090](#). S. Ferrara, G. W. Gibbons and R. Kallosh, *Black Holes and Critical Points in Moduli Space*, Nucl. Phys. **B500** (1997) 75, [hep-th/9702103](#).
- [9] S. Ferrara and M. Günaydin, *Orbits of exceptional groups, duality and BPS states in string theory*, Int. J. Mod. Phys. **A13**, 2075 (1998), [hep-th/9708025](#). H. Lu, C.N. Pope, K.S. Stelle, *Multiplet structures of BPS solitons*, Class. Quant. Grav. **15**, 537 (1998), [hep-th/9708109](#).
 - [10] L. Borsten, M. J. Duff, S. Ferrara, A. Marrani, W. Rubens, *Small Orbits*, Phys. Rev. **D85**, 086002 (2012), [arXiv:1108.0424 \[hep-th\]](#).
 - [11] S. Ferrara and A. Marrani, *On the Moduli Space of non-BPS Attractors for $\mathcal{N}=2$ Symmetric Manifolds*, Phys. Lett. **B652**, 111 (2007), [arXiv:0706.1667](#).
 - [12] E. A. Bergshoeff, A. Marrani, F. Riccioni, *Brane orbits*, Nucl. Phys. **B861**, 104 (2012), [arXiv:1201.5819 \[hep-th\]](#).
 - [13] M. Günaydin, G. Sierra, P. K. Townsend, *Exceptional Supergravity Theories and the Magic Square*, Phys. Lett. **B133**, 72 (1983). M. Günaydin, G. Sierra and P. K. Townsend, *The Geometry of $\mathcal{N}=2$ Maxwell-Einstein Supergravity and Jordan Algebras*, Nucl. Phys. **B242**, 244 (1984).
 - [14] V. K. Dobrev, *Exceptional Lie Algebra $E_{7(-25)}$: Multiplets and Invariant Differential Operators*, J.Phys. **A42**, 285203 (2009), [arXiv:0812.2690 \[hep-th\]](#).
 - [15] K. Holland, P. Minkowski, M. Pepe and U. J. Wiese, *Exceptional confinement in G_2 gauge theory*, Nucl. Phys. **B668**, 207 (2003), [hep-lat/0302023](#).
 - [16] J.P. Keating, N.Linden and Z. Rudnick, *Random Matrix Theory, The exceptional Lie groups, and L-functions*, J. Phys. **A36** no. 12, 2933 (special RMT volume) (2003).
 - [17] W. Krauth and M. Staudacher, *Yang-Mills integrals for orthogonal, symplectic and exceptional groups*, Nucl. Phys. **B584**, 641 (2000), [hep-th/0004076](#).
 - [18] G. Cossu, M. D’Elia, A. Di Giacomo, B. Lucini, C. Pica, *Confinement: G_2 group case*, PoSLAT2007, 296 (2007), [arXiv:0710.0481 \[hep-lat\]](#).
 - [19] J. C. Baez, *The Octonions*, Bull. Am. Math. Soc. **39**, 145 (2002), [math/0105155](#).
 - [20] A. Anastasiou, L. Borsten, M.J. Duff, L.J. Hughes, S. Nagy, *Super Yang-Mills, division algebras and triality*, [arXiv:1309.0546 \[hep-th\]](#). A. Anastasiou, L. Borsten, M.J. Duff, L.J. Hughes, S. Nagy, *A magic pyramid of supergravities*, [arXiv:1312.6523 \[hep-th\]](#).
 - [21] F. Gürsey, in : “*First workshop on Grand Unification*”, P. Frampton, S. H. Glashow, A. Yildiz Eds. (Math. Sci. Press, 1980).
 - [22] F. Gürsey, P. Ramond, P. Sikivie, *A Universal Gauge Theory Model Based on E_6* , Phys. Lett. **B60**, 177 (1976). F. Gürsey and P. Sikivie, *E_7 as a Universal Gauge Group*, Phys. Rev. Lett. **36**, 775 (1976).
 - [23] F. Caravaglios and S. Morisi, *Gauge boson families in grand unified theories of fermion masses: $E_6^4 \times S_4$* , Int. J. Mod. Phys. **A22**, 2469 (2007), [hep-ph/0611068](#). F. Caravaglios and S. Morisi, *Fermion masses in E_6 grand unification with family permutation symmetries*, [hep-ph/0510321](#). C. R. Das and L. V. Laperashvili, *Preon model and family replicated E_6 unification*, SIGMA **4**, 012 (2008), [arXiv:0707.4551 \[hep-ph\]](#).

- [24] R. B. Brown, *Groups of type E_7* , J. Reine Angew. Math. **236**, 79 (1969).
- [25] S. Ferrara and R. Kallosh, *Creation of Matter in the Universe and Groups of Type E_7* , JHEP **1112**, 096 (2011), [arXiv:1110.4048 \[hep-th\]](#). S. Ferrara, R. Kallosh, and A. Marrani, *Degeneration of Groups of Type E_7 and Minimal Coupling in Supergravity*, JHEP **1206**, 074 (2012), [arXiv:1202.1290 \[hep-th\]](#).
- [26] A. Marrani, C.-X. Qiu, S.-Y. D. Shih, A. Tagliaferro, B. Zumino, *Freudenthal Gauge Theory*, JHEP **1303**, 132 (2013), [arXiv:1208.0013 \[hep-th\]](#).
- [27] L. Borsten, M.J. Duff, A. Marrani, W. Rubens, *On the Black-Hole/Qubit Correspondence*, Eur. Phys. J. Plus **126**, 37 (2011), [arXiv:1101.3559 \[hep-th\]](#). L. Borsten, M.J. Duff, P. Lévy, *The black-hole/qubit correspondence: an up-to-date review*, Class. Quant. Grav. **29**, 224008 (2012), [arXiv:1206.3166 \[hep-th\]](#).
- [28] N. Marcus and J.H. Schwarz, *Three-dimensional supergravity theories*, Nuclear Phys. **B228** (1983), 145.
- [29] D.J. Gross, J.A. Harvey, E. Martinec, and R. Rohm, *Heterotic string*, Physical Review Letters **54**, 502 (1985).
- [30] D. Vogan, *The character table for E_8* , Notices of the AMS **54** (2007), no. 9, 1022.
- [31] A. Garrett Lisi, *An exceptionally simple theory of everything*, [arXiv:0711.0770 \[hep-th\]](#).
- [32] J. Distler and S. Garibaldi, *There is no “Theory of Everything” inside E_8* , Comm. Math. Phys. **298** (2010), 419, [arXiv:0905.2658 \[math.RT\]](#).
- [33] R. Coldea, D.A. Tennant, E.M. Wheeler, E. Wawrzynska, D. Prabhakaran, M. Telling, K. Habicht, P. Smid, and K. Kiefer, *Quantum criticality in an Ising chain: experimental evidence for emergent E_8 symmetry*, Science **327**, 177 (2010). D. Borthwick and S. Garibaldi, *Did a 1-dimensional magnet detect a 248-dimensional Lie algebra?*, Not. Amer. Math. Soc. **58**, 1055 (2011), [arXiv:1012.5407 \[math-ph\]](#).
- [34] C. Rovelli, *“Quantum Gravity”* (Cambridge University Press, 2004).
- [35] A. Garrett Lisi, L. Smolin, S. Speziale, *Unification of gravity, gauge fields and Higgs bosons*, J.Phys. **A43**, 445401 (2010), [arXiv:1004.4866 \[gr-qc\]](#).
- [36] P. Truini, *Exceptional Lie Algebras, $SU(3)$ and Jordan Pairs*, Pacific J. Math. **260**, 227 (2012), [arXiv:1112.1258 \[math-ph\]](#).
- [37] K. McCrimmon : *“A Taste of Jordan Algebras”* (Springer-Verlag New York Inc., New York, 2004).
- [38] J. M. Landsberg and L. Manivel, *Triality, Exceptional Lie Algebras and Deligne Dimension Formulas*, Adv. Math. **15**, 59 (2002), [arXiv:math/0107032](#).
- [39] S. Ferrara, A. Marrani, B. Zumino, *Jordan Pairs, E_6 and U-Duality in Five Dimensions*, J. Phys. **A42**, 065402 (2013), [arXiv:1208.0347 \[math-ph\]](#).
- [40] M. Günaydin, K. Koepsell and H. Nicolai, *Conformal and Quasiconformal Realizations of Exceptional Lie Groups*, Commun. Math. Phys. **221**, 57 (2001), [hep-th/0008063](#).

- [41] M. Günaydin and O. Pavlyk, *Minimal Unitary Realizations of Exceptional U-Duality Groups and their Subgroups as Quasiconformal Groups*, JHEP **0501**, 019 (2005), [hep-th/0409272](#).
- [42] M. Günaydin, *Lectures on Spectrum Generating Symmetries and U-duality in Supergravity, Extremal Black Holes, Quantum Attractors and Harmonic Superspace*, [arXiv:0908.0374 \[hep-th\]](#).
- [43] S. Ferrara, A. Marrani, M. Trigiante, *Super-Ehlers in Any Dimension*, JHEP **1211**, 068 (2012), [arXiv:1206.1255 \[hep-th\]](#).
- [44] S. L. Cacciatori, B. L. Cerchiai, A. Della Vedova, G. Ortenzi and A. Scotti, *Euler angles for G_2* , J. Math. Phys. **46**, 083512 (2005), [hep-th/0503106](#). S. L. Cacciatori, *A simple parametrization for G_2* , J. Math. Phys. **46**, 083520 (2005), [math-ph/0503054](#). F. Bernardoni, S. L. Cacciatori, B. L. Cerchiai, A. Scotti, *Mapping the geometry of the F_4 group*, Adv. Theor. Math. Phys. **12**, 889 (2008), [arXiv:0705.3978](#). F. Bernardoni, S. L. Cacciatori, B. L. Cerchiai, A. Scotti, *Mapping the geometry of the E_6 group*, J. Math. Phys. **49**, 012107 (2008), [arXiv:0710.0356](#). S. L. Cacciatori, B. L. Cerchiai, *Exceptional groups, symmetric spaces and applications*, in : “Group Theory : Classes, Representations, Connections, and Applications”, 177 (2010), C. W. Danellis Ed. (Nova Science Publ., New York, 2010), [arXiv:0906.0121 \[math-ph\]](#). S. L. Cacciatori, F. Dalla Piazza, A. Scotti, *E_7 groups from octonionic magic square*, [arXiv:1007.4758 \[math-ph\]](#). S. L. Cacciatori, F. Dalla Piazza, A. Scotti, *A simple E_8 construction*, [arXiv:1207.3623 \[math-ph\]](#).
- [45] O.Loos : “*Jordan Pairs*”, Lect. Notes Math. **460**, (Springer, 1975).
- [46] P. Jordan, J. von Neumann, E. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, Ann. Math. **35**, 29 (1934).
- [47] N. Jacobson, *Structure Theory for a class of Jordan Algebras*, Proc. Nat. Acad. Sci. U.S.A., **55** 243 (1966).
- [48] K. Meyberg, *Jordan-Triplesysteme und die Koecher-Konstruktion von Lie Algebren*, Math. Z. **115**, 58 (1970).
- [49] J. Tits, *Une classe d’algèbres de Lie en relation avec les algèbres de Jordan*, Nederl. Akad. Wetensch. Proc. Ser. **A 65** = Indagationes Mathematicae **24**, 530 (1962).
- [50] I. L. Kantor, *Classification of irreducible transitive differential groups*, Doklady Akademiii Nauk SSSR **158**, 1271 (1964).
- [51] M.Koecher, *Imbedding of Jordan algebras into Lie algebras. I.*, Am. J. Math. **89**, 787 (1967).
- [52] J.R. Faulkner, *Jordan pairs and Hopf algebras*, J. of Algebra, **232**, 152 (2000).
- [53] R. D. Schafer: “*An Introduction to Non Associative Algebras*” (Academic Press, 1966).
- [54] R. D. Schafer, *Inner derivations of non associative algebras*, Bull. Amer. Math. Soc. **55**, 769 (1949).
- [55] M. Zorn, *Alternativkörper und quadratische systeme*, Abh. Math. Sem. Univ. Hamburg **9**, 395 (1933).
- [56] O. Loos, H.P. Petersson, M.L. Racine, *Inner derivations of alternative algebras over commutative rings*, Algebra & Number Theory **2**, 927 (2008).

- [57] J.Tits, *Algèbres alternatives, algèbres de Jordan et algèbres de Lie exceptionnelles, I Construction*, Nederl. Akad. Wetensch. Proc. Ser. **A 69**, 223 (1966).
- [58] H. Freudenthal, *Beziehungen der E_7 und E_8 zur Oktavenebene V-IX*, Proc. K. Ned. Akad. Wet. **A 62**, 447 (1959).
- [59] N. Jacobson: “*Exceptional Lie Algebras*”, Lecture Notes in Pure and Applied Mathematics **1** (M. Dekker, 1971).
- [60] K. Imaeda and M. Imaeda, *Sedenions: algebra and analysis*, Appl. Math. and Comp. **115**, 77 (2000). G. Moreno, *The zero divisors of the Cayley–Dickson algebras over the real numbers*, Sociedad Matemática Mexicana, Boletín Tercera Serie **4** (1), [q-alg/9710013](#).
- [61] M. J. Duff and S. Ferrara, *E_6 and the bipartite entanglement of three qutrits*, Phys. Rev. **D76**, 124023 (2007), [arXiv:0704.0507 \[hep-th\]](#).
- [62] T. Kugo and P. Townsend, *Supersymmetry and the division algebras*, Nuclear Physics, Section **B221**, 357 (1983).
- [63] M. Rios, *Extremal Black Holes as Qudits*, [arXiv:1102.1193 \[hep-th\]](#).
- [64] B. W. Westbury, *Sextonions and the magic square*, J. London Math. Soc. **73**, 455 (2006), [math.RA/0411428](#).
- [65] J.M. Landsberg and L. Manivel, *The sextonions and $E_{7\frac{1}{2}}$* , Adv. Math. **201**, 143 (2006), [math/0402157](#).